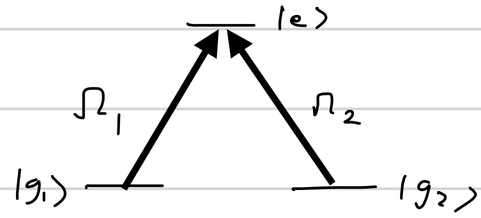
Physics 566: Quantum Optics I
Problem Set #6: Solutions

Problem 2: Dark States

We consider the 3-level "lambda system" with two ground states resonantly coupled to an excited state.



In the rotating frame: $\hat{H}_{RF} = \frac{\hbar\Omega_1}{2}\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \frac{\hbar\Omega_2}{2}\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)$

(a) The "dressed states" are the eigenstates of the coupled atom-laser system

The Hamiltonian matrix: $\hat{H}_{RF} = \frac{\hbar}{2}\begin{bmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{bmatrix}$ In the ordered basis $\{|g_1\rangle, |g_2\rangle, |e\rangle\}$

The eigenvectors / eigenvalues of this matrix are:

$$|\psi_{Dark}\rangle = \frac{-\Omega_2|1\rangle + \Omega_1|2\rangle}{\sqrt{\Omega_1^2 + \Omega_2^2}}, \qquad E_{Dark} = 0 \qquad \text{(Dark state)}$$

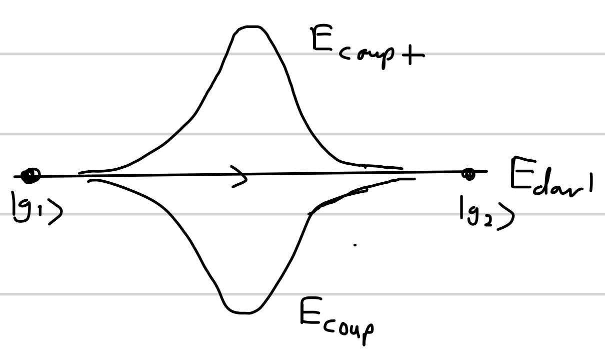
$$|\psi_{Coup,\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{Bright}\rangle \pm |e\rangle\right), \qquad E_{coup,\pm} = \pm\frac{\hbar}{2}\sqrt{\Omega_1^2+\Omega_2^2} \qquad \text{(Coupled states)}$$

$|\psi_{Bright}\rangle = (\Omega_1|1\rangle + \Omega_2|2\rangle)/\sqrt{\Omega_1^2+\Omega_2^2}$

(b) One can transfer population from $|g_1\rangle$ to $|g_2\rangle$ using the "counter-intuitive pulse sequence".

First apply field that couples $|2\rangle \to |3\rangle$. As we turn that field off turn on $|1\rangle \to |3\rangle$

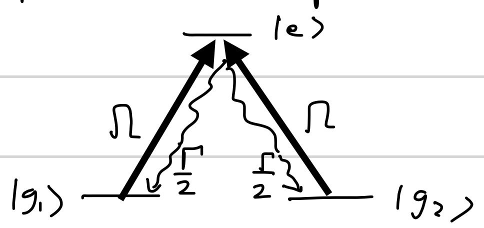
The counter-intuitive pulse sequence achieves the transfer from $|g_1\rangle \to |g_2\rangle$ by adiabatic following of the dark state. The dressed eigenvalues as a function of time appear as:



At $t=0$ $|\psi\rangle = |g_1\rangle$. As we slowly turn on Ω_2, $|\psi\rangle$ is in the dark state (the eigenstate of $\hat{H}$ with $\Omega_1 = 0$)). If we turn on and off the fields slowly, so the transition is adiabatic, then $|\psi(t)\rangle$ follows $|\psi_{Dark}(t)\rangle = \dfrac{\Omega_2(t)|2\rangle - \Omega_1(t)|1\rangle}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}}$. Intermediately, when

$\Omega_1 = \Omega_2$ $|\psi_{Dark}\rangle = \dfrac{|1\rangle - |2\rangle}{\sqrt{2}}$. At the end of pulses $|\psi(t)\rangle = |\psi_{Dark}(t \to \infty)\rangle = |2\rangle$.

(c) We now consider spontaneous decay. We simplify to the case $\Omega_1 = \Omega_2 = \Omega$; $\Gamma_{31} = \Gamma_{32} = \frac{\Gamma}{2}$



In the basis $\{|D\rangle, |B\rangle, |e\rangle\}$ where $|D\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$, $|B\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$, $|e\rangle$

$|1\rangle = \frac{1}{\sqrt{2}}(|D\rangle + |B\rangle)$; $|2\rangle = \frac{1}{\sqrt{2}}(|B\rangle - |D\rangle)$)

$$\Rightarrow \hat{H} = \frac{\sqrt{2}\hbar\Omega}{2}\left(|B\rangle\langle e| + |e\rangle\langle B|\right)$$

(effective 2-level coupling between $|B\rangle$ & $|e\rangle$ w/ Rabi frequency $\sqrt{2}\Omega$
The Dark State is decoupled)

the effective Hamiltonian $\hat{H}_{eff} = \hat{H} - i\frac{\hbar}{2}\Gamma |e\rangle\langle e|$

The "Feeding term" in the Master Equation: $\mathcal{L}_{feed}[\hat{\rho}] = \frac{\Gamma}{2}\rho_{ee}(|1\rangle\langle 1| + |2\rangle\langle 2|) = \frac{\Gamma}{2}\rho_{ee}(|B\rangle\langle B| + |D\rangle\langle D|)$

(d) The Master equation: $\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{+}\right) + \mathcal{L}_{feed}[\hat{\rho}]$

Non-Hermitian Schrödinger Eqn: $\frac{\partial}{\partial t}|\psi\rangle = -\frac{i}{\hbar}\hat{H}_{eff}|\psi\rangle$

$\Rightarrow \dot{c}_e = -\frac{\Gamma}{2}c_e - i\frac{\sqrt{2}\Omega}{2}c_B$, $\dot{c}_B = -i\frac{\sqrt{2}\Omega}{2}c_e$, $\dot{c}_D = 0$

$$\boxed{\begin{aligned}
\Rightarrow \dot{\rho}_{ee} &= \dot{c}_e c_e^* + c_e \dot{c}_e^* = -\Gamma\rho_{ee} - i\sqrt{2}\frac{\Omega}{2}(\rho_{Be} - \rho_{eB}) \\
\dot{\rho}_{BB} &= \dot{c}_B c_B^* + \dot{c}_B^* c_B + \langle B|\mathcal{L}_{feed}[\hat{\rho}]|B\rangle = +\frac{\Gamma}{2}\rho_{ee} - i\frac{\sqrt{2}\Omega}{2}(\rho_{eB} - \rho_{Be}) \\
\dot{\rho}_{DD} &= \dot{c}_D c_D^* + c_D\dot{c}_D^* + \langle D|\mathcal{L}_{feed}[\hat{\rho}]|D\rangle = \frac{\Gamma}{2}\rho_{ee} \\
\dot{\rho}_{eB} &= \dot{c}_e c_B^* + c_e\dot{c}_B^* = -\frac{\Gamma}{2}\rho_{eB} + i\frac{\sqrt{2}\Omega}{2}(\rho_{ee} - \rho_{BB}) \\
\dot{\rho}_{eD} &= \dot{c}_e c_D^* + c_e\dot{c}_D^* = -\frac{\Gamma}{2}\rho_{eD} - i\frac{\sqrt{2}\Omega}{2}\rho_{BD} \\
\dot{\rho}_{BD} &= \dot{c}_B c_D^* + c_B\dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2}\rho_{eD}
\end{aligned}}$$

The steady-state solution: $\dot{\rho}_{BB} = \frac{\Gamma}{2}\rho_{ee} = 0 \Rightarrow \rho_{ee} = 0$

$$\Rightarrow \dot{\rho}_{ee} = i\frac{\sqrt{2}\Omega}{2}(\rho_{eB} - \rho_{Be}) = 0 \Rightarrow \mathrm{Im}(\rho_{eB}) = 0$$

$\Rightarrow \dot{\rho}_{eB} = -\frac{\Gamma}{2}\rho_{eB} - i\frac{\sqrt{2}\Omega}{2}\rho_{BB} = 0 \Rightarrow \mathrm{Re}(\rho_{eB}) = 0$ and $\rho_{BB} = 0$ (since Real & Imag $= 0$)

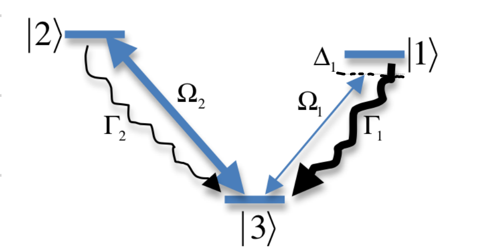
$$\left.\begin{aligned} \dot{\rho}_{eD} &= \dot{c}_e c_D^* + c_e \dot{c}_D^* = -\frac{\Gamma}{2}\rho_{eD} + i\frac{\sqrt{2}\Omega}{2}\rho_{BD} = 0 \\ \dot{\rho}_{BD} &= \dot{c}_B c_D^* + c_B \dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2}\rho_{eD} = 0 \end{aligned}\right\} \Rightarrow \rho_{eD} = 0,\ \rho_{BD} = 0$$

$\Rightarrow$ Dynamically ALL elements of $\hat{\rho} \to 0$ in steady state except ρ_{DD}

$\Rightarrow$ Since the equation is trace preserving, in steady state, $\hat{\rho}_{s.s.} = |D\rangle\langle D|$

Problem 3: Autler-Townes Splitting

We consider a V-configuration:



A strong field on the $|2\rangle \leftrightarrow |3\rangle$ transition "dresses" these states. We then probe these dressed states with a weak probe on the $|3\rangle \leftrightarrow |1\rangle$ transition and measure the absorption as a function of detuning, Δ_1.

(a) The (trace-preserving) master equation for this 3-level atom is:

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{\dagger}\right) + \left(\Gamma_1 \rho_{11} + \Gamma_2 \rho_{22}\right)|3\rangle\langle 3|$$

where $\hat{H}_{eff} = -\hbar\left(\Delta_1 + i\frac{\Gamma_1}{2}\right)|1\rangle\langle 1| - \frac{i}{2}\hbar\Gamma_2 |2\rangle\langle 2| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right)$

Non-unitary evolution of probability amplitudes: $\frac{d}{dt}c_\alpha = -\frac{i}{\hbar}\langle\alpha|\hat{H}_{eff}|\psi\rangle$, $|\psi\rangle = \sum_{\alpha=1}^{3} c_\alpha |\alpha\rangle$

$$\Rightarrow \dot{c}_1 = \left(i\Delta_1 - \frac{\Gamma_1}{2}\right)c_1 - i\frac{\Omega_1}{2}c_3, \quad \dot{c}_2 = -\frac{\Gamma_2}{2}c_2 - i\frac{\Omega_2}{2}c_3, \quad \dot{c}_3 = -i\frac{\Omega_1}{2}c_1 - i\frac{\Omega_2}{2}c_2$$

The Master equation: $\dot{\rho}_{\alpha\beta} = \dot{c}_\alpha c_\beta^* + c_\alpha \dot{c}_\beta^* + \left(\Gamma_1\rho_{11} + \Gamma_2\rho_{22}\right)\delta_{\alpha 3}\delta_{\beta 3}$

$$\Rightarrow \dot{\rho}_{11} = \dot{c}_1 c_1^* + c_1\dot{c}_1^* = -\Gamma_1\rho_{11} - i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right), \quad \dot{\rho}_{22} = \dot{c}_2 c_2^* + c_2\dot{c}_2^* = -\Gamma_2\rho_{22} - i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right)$$

$$\dot{\rho}_{33} = -\dot{\rho}_{11} - \dot{\rho}_{22} = +\left(\Gamma_1\rho_{11} + \Gamma_2\rho_{22}\right) + i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right) + i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right)$$

$$\dot{\rho}_{13} = \dot{c}_1 c_3^* + c_1\dot{c}_3^* = \left(i\Delta_1 - \frac{\Gamma_1}{2}\right)\rho_{13} - i\frac{\Omega_1}{2}\left(\rho_{33} - \rho_{11}\right) + i\frac{\Omega_2}{2}\rho_{12}$$

$$\dot{\rho}_{23} = \dot{c}_2 c_3^* + c_2\dot{c}_3^* = -\frac{\Gamma_2}{2}\rho_{23} - \frac{i}{2}\Omega_2\left(\rho_{33} - \rho_{22}\right) + i\frac{\Omega_1}{2}\rho_{21}$$

$$\dot{\rho}_{12} = \dot{c}_1 c_2^* + c_1\dot{c}_2^* = \left(i\Delta_1 - \frac{\Gamma_1 + \Gamma_2}{2}\right)\rho_{12} + \frac{i}{2}\Omega_2\rho_{13} - \frac{i}{2}\Omega_1\rho_{32}$$

(b) Under the assumption of weak excitation of $|1\rangle$, retain terms only linear in Ω_1.

The state state solutions, $\dot{\rho}_{\alpha\beta} = 0$

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$$-\Gamma_1\rho_{11} - \frac{i}{2}\Omega_1(\rho_{31}-\rho_{13}) = 0 \qquad \Rightarrow \rho_{11} = \frac{\Omega_1}{\Gamma_1}\left(\frac{\rho_{13}-\rho_{31}}{2i}\right) = \frac{\Omega_1}{\Gamma}\mathrm{Im}(\rho_{13})$$

$$\left(i\Delta_1 - \frac{\Gamma_1}{2}\right)\rho_{13} - \frac{i\Omega_1}{2}\left(\rho_{33} - \underbrace{\cancel{\rho_{11}}}_{\mathcal{O}(\Omega_1)}\right) + \frac{i}{2}\Omega_2\rho_{12} \qquad \Rightarrow \rho_{13} \approx \frac{-\Omega_1}{(2\Delta_1+i\Gamma_1)}\rho_{33} + \frac{\Omega_2}{(2\Delta_1+i\Gamma_1)}\rho_{21}$$

$$\left(i\Delta_1 - \frac{\Gamma}{2}\right)\rho_{12} + \frac{i}{2}\Omega_2\rho_{13} - \frac{i}{2}\underbrace{\cancel{\Omega_1\rho_{32}}}_{\mathcal{O}(\Omega_1^2)} \qquad \Rightarrow \rho_{12} \approx \left(\frac{\Omega_2}{2\Delta_1+i\Gamma_1}\right)\rho_{13}$$

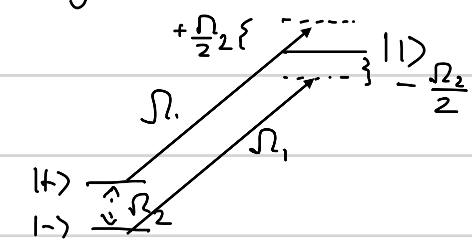
(c) Putting this all together:

$$\rho_{13} \approx \frac{-\Omega_1}{2\Delta_1+i\Gamma_1}\rho_{33} + \frac{\Omega_2^2}{(2\Delta_1+i\Gamma_1)^2}\rho_{13} \Rightarrow \rho_{13} \approx \frac{-\Omega_1(2\Delta_1+i\Gamma_1)}{(2\Delta_1+i\Gamma_1)^2-\Omega_2^2}\rho_{33}$$

$$\Rightarrow \boxed{\rho_{11} = -\frac{\Omega_1^2}{\Gamma_1^2}\rho_{33}\,\mathrm{Im}\left[\frac{2\Delta_1+i\Gamma_1}{(2\Delta_1+i\Gamma_1)^2-\Omega_2^2}\right]}$$

(d) The Autler-Twe splitting is understood in the Dressed-basis of the strongly-coupled $|2\rangle - |3\rangle$ system:

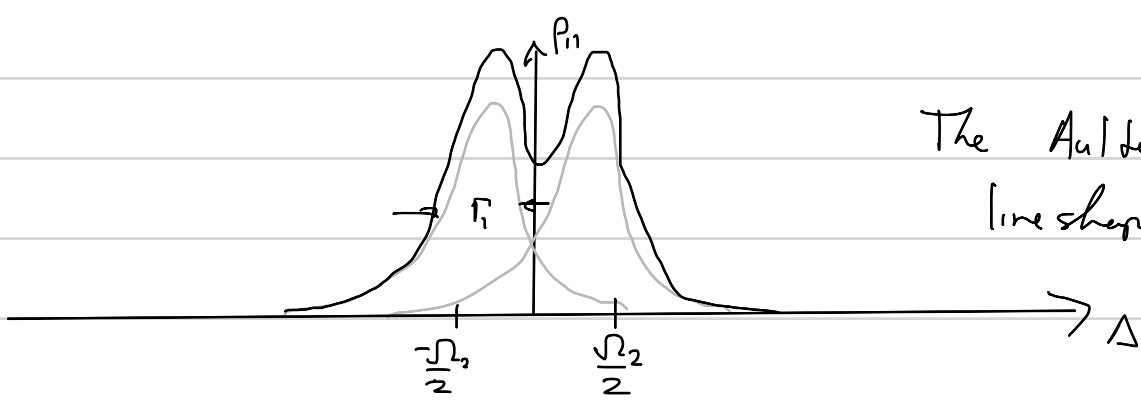
Sketch for $\Delta_1 = 0$



There are two resonances $|+\rangle \to |1\rangle$, $|-\rangle \to |2\rangle$

$$|\pm\rangle = \frac{|2\rangle \pm |3\rangle}{\sqrt{2}}$$

These two resonances are split by the dressed state splitting $\pm\frac{\Omega_2}{2}$. The width of each resonance is Γ_1.



The Autler Townes line shape

(c) The difference between EIT and Autler-Townes splitting is that the former is an interference effect arising from the destructive interference of two paths, whereas Autler-Townes is the incoherent sum of two distinguishable paths. In EIT the absorption goes exactly to zero at $\Delta_1 = 0$ whereas in Autler-Townes, it is reduced.

When $\Omega_2 \gg \Gamma_1$, the two paths of EIT are essentially distinguishable — EIT and Autler-Townes absorption spectra look essentially the same in this situation, i.e., two separated Lorentzians, with essentially zero absorption between.