

Phys 566, Quantum Optics

Problem Set #7 Solutions

Problem 1: Momentum and Angular Momentum in Field

$$\hat{\vec{p}} = \int d^3x \left(\frac{\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x})}{4\pi c} \right), \quad \hat{\vec{J}} = \int d^3x \left(\vec{x} \times \frac{\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x})}{4\pi c} \right)$$

Quantized field

$$\hat{\vec{A}} = \underbrace{\sum_{\vec{k}, \lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \vec{e}_{\vec{k}, \lambda} \hat{a}_{k, \lambda} e^{i\vec{k} \cdot \vec{x}}}_{\hat{\vec{A}}^{(+)}} + \underbrace{h.c.}_{\hat{\vec{A}}^{(-)}}$$

$$\hat{\vec{E}} = \underbrace{+i \sum_{\vec{k}, \lambda} \sqrt{\frac{2\pi\hbar\omega_k}{V}} \vec{e}_{\vec{k}, \lambda} e^{i\vec{k} \cdot \vec{x}} \hat{a}_{k, \lambda}}_{\hat{\vec{E}}^{(+)}} + \underbrace{h.c.}_{\hat{\vec{E}}^{(-)}}$$

$$\hat{\vec{B}} = \underbrace{+i \sum_{\vec{k}, \lambda} \sqrt{\frac{2\pi\hbar c}{V}} \hat{k} \times \vec{e}_{\vec{k}, \lambda} e^{i\vec{k} \cdot \vec{x}} \hat{a}_{k, \lambda}}_{\hat{\vec{B}}^{(+)}} + \underbrace{h.c.}_{\hat{\vec{B}}^{(-)}}$$

Here I have used the basis of circularly polarized plane waves with orthogonality

$$\vec{e}_{\vec{k}, \lambda}^* \cdot \vec{e}_{\vec{k}', \lambda'} = \delta_{\vec{k} \vec{k}'} \delta_{\lambda \lambda'}$$

Note also $\vec{e}_{-\vec{k}, \lambda_1} = \vec{e}_{\vec{k}, \lambda_2}$

(1a)

Let us plug the mode decomposition into $\hat{\vec{p}}$

$$\Rightarrow \hat{\vec{p}} = \int d^3x \left(\frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} + \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(+)}}{4\pi c} + h.c. \right)$$

Consider first term:

$$\begin{aligned} \int d^3x \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} &= \sum_{\vec{k}, \lambda, \vec{k}', \lambda'} \frac{1}{4\pi c} (2\pi\hbar\sqrt{\omega_k\omega_{k'}}) \vec{E}_{\vec{k}, \lambda} \times (\vec{k}' \times \vec{E}_{\vec{k}', \lambda'}^*) \\ &\quad \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}}}_{= \delta_{\vec{k}\vec{k}'}^{(3)}} \hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}', \lambda'}^\dagger \\ &\Rightarrow \sum_{\vec{k}} \sum_{\lambda, \lambda'} \frac{\hbar k}{2} \left[\underbrace{\vec{E}_{\vec{k}, \lambda} \times (\vec{k} \times \vec{E}_{\vec{k}, \lambda}^*)}_{\hbar(\vec{E}_{\vec{k}, \lambda} \cdot \vec{E}_{\vec{k}, \lambda}^*) - \vec{E}_{\vec{k}, \lambda}^* (\vec{k} \cdot \vec{E}_{\vec{k}, \lambda})} \right] \hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda}^\dagger \\ &\quad = 0 \quad (\text{transversality}) \end{aligned}$$

$$\Rightarrow \int d^3x \frac{\vec{E}^{(+)} \times \vec{B}^{(-)}}{4\pi c} = \frac{1}{2} \sum_{\vec{k}, \lambda} \hbar \vec{k} \hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda}^\dagger$$

Now, by similar steps

$$\begin{aligned} \int d^3x \frac{\vec{E}^{(+)} \times \vec{B}^{(+)}}{4\pi c} &= \sum_{\vec{k}} \sum_{\lambda, \lambda'} \frac{\hbar k}{2} \left(\underbrace{\vec{E}_{\vec{k}, \lambda} \times (-\vec{k} \times \vec{E}_{-\vec{k}, \lambda}^*)}_{-\vec{k}(\vec{E}_{\vec{k}, \lambda} \cdot \vec{E}_{-\vec{k}, \lambda}^*) + \vec{E}_{-\vec{k}, \lambda}^* (\vec{k} \cdot \vec{E}_{\vec{k}, \lambda})} \right) \\ &\quad = 0 \quad (\text{see page 4}) \end{aligned}$$

Thus,

$$\hat{\vec{p}} = \int d^3x \frac{\vec{E}^{(+)} \times \vec{B}^{(-)}}{4\pi c} + h.c.$$

$$= \sum_{\vec{k}, \lambda} \hbar \vec{k} \left(\frac{\hat{a}_{\vec{k}, \lambda} \hat{a}_{\vec{k}, \lambda}^{\dagger} + \hat{a}_{\vec{k}, \lambda}^{\dagger} \hat{a}_{\vec{k}, \lambda}}{2} \right)$$

$$= \sum_{\vec{k}, \lambda} \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}^{\dagger} \hat{a}_{\vec{k}, \lambda} + \frac{\hbar}{2} \sum_{\lambda} \left(\sum_{\vec{k}} \vec{k} \right) \rightarrow 0$$

no
mean
momentum

$$\Rightarrow \boxed{\hat{\vec{p}} = \sum_{\vec{k}, \lambda} \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}^{\dagger} \hat{a}_{\vec{k}, \lambda}}$$

Neat! in the plane wave decomposition, we see that the momentum in the field decompose into sum of photon momenta $\hbar \vec{k}$ times the number operator

$$\hat{N}_{\vec{k}} = \sum_{\lambda} \hat{a}_{\vec{k}, \lambda}^{\dagger} \hat{a}_{\vec{k}, \lambda} \quad \text{counting the}$$

number of photons with wave vector $\vec{k}$.

(b) Total angular momentum in field:

$$\hat{\vec{J}} = \int d^3x \vec{x} \times \hat{\vec{P}}(\vec{x})$$

where $\hat{\vec{P}}(\vec{x}) = \frac{1}{4\pi c} (\hat{\vec{E}} \times \hat{\vec{B}}) =$ momentum density

Lets massage these equations a bit.

$$(\hat{\vec{E}} \times \hat{\vec{B}})_i = \epsilon_{ijk} E_j B_k \quad (\text{summation convention})$$

$$= \epsilon_{ijk} E_j \epsilon_{k\ell m} \partial_\ell A_m$$

$$= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) E_j \partial_\ell A_m$$

$$= E_\ell \partial_i A_\ell - E_\ell \partial_\ell A_i$$

Now $\vec{J} = \int d^3x \vec{x} \times \vec{P}(\vec{x})$

$$\Rightarrow J_j = \epsilon_{jki} \int d^3x x_k P_i$$

$$= \epsilon_{jki} \frac{1}{4\pi c} \int d^3x [E_\ell (x_k \partial_i) A_\ell - (x_k E_\ell) (\partial_\ell A_i)]$$

$$= \frac{1}{4\pi c} \int d^3x E_\ell (\vec{x} \times \vec{\nabla})_j A_\ell$$

$$+ \frac{1}{4\pi c} \int d^3x \underbrace{\epsilon_{jki} \partial_\ell (x_k E_\ell)}_{[\delta_{\ell k} + \vec{\nabla} \times \vec{E}]} A_i$$

(integration by parts)

$\rightarrow 0$ in free space

$$\Rightarrow \vec{J}_j = \frac{1}{4\pi c} \left(\int d^3x \vec{E}_\ell (\vec{x} \times \vec{\nabla})_j A_\ell + \int d^3x (\vec{E} \times \vec{A})_j \right)$$

$$\Rightarrow \vec{J} = \vec{J}_{\text{orb}} + \vec{J}_{\text{spin}}$$

$$\begin{aligned} \vec{J}_{\text{orb}} &= \frac{1}{4\pi c} \int d^3x \vec{E}_\ell (\vec{x} \times \vec{\nabla}) A_\ell \\ \vec{J}_{\text{spin}} &= \frac{1}{4\pi c} \int d^3x (\vec{E} \times \vec{A}) \end{aligned}$$

(1c)

Let us expand these terms in the plane wave basis:

$$\begin{aligned} \vec{J}_{\text{orb}} &= \left(\frac{1}{4\pi c} \int d^3x \vec{E}_\ell^{(-)} (\vec{x} \times \vec{\nabla}) A_\ell^{(+)} + \text{h.c.} \right) \\ &\quad + \left(\frac{1}{4\pi c} \int d^3x \vec{E}_\ell^{(+)} (\vec{x} \times \vec{\nabla}) \vec{A}_\ell^{(-)} + \text{h.c.} \right) \end{aligned}$$

Consider

$$\frac{1}{4\pi c} \int d^3x \vec{E}_\ell^{(-)} (\vec{x} \times \vec{\nabla}) A_\ell^{(+)}$$

$$= \sum_{\vec{k}, \lambda, \lambda'} \frac{\hbar}{2} \sqrt{\frac{\omega_{\vec{k}}}{\omega_{\vec{k}'}}} \hat{a}_{\vec{k}, \lambda'}^\dagger \hat{a}_{\vec{k}, \lambda} \vec{E}_{\vec{k}, \lambda}^* \cdot \vec{E}_{\vec{k}, \lambda'}$$

(Summing over ℓ)

$$\underbrace{\int \frac{d^3x}{V} e^{-i\vec{k}' \cdot \vec{x}} (\vec{x} \times -i\vec{\nabla}) e^{i\vec{k} \cdot \vec{x}}}_{\mathcal{I}}$$

Aside: $\mathcal{L} = \int \frac{d^3x}{V} e^{-i\vec{k}' \cdot \vec{x}} (\vec{x} \times -i\vec{\nabla}) e^{i\vec{k} \cdot \vec{x}}$

$$= \int \frac{d^3x}{V} e^{-i\vec{k}' \cdot \vec{x}} (\vec{x} \times \vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

$$= \left[\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} \vec{x} \right] \times \vec{k}$$

$$= +i\vec{\nabla}_{\vec{k}'} \left[\underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}}}_{\delta_{\vec{k}, \vec{k}'}} \right] \times \vec{k}$$

$$\Rightarrow \frac{1}{4\pi c} \int d^3x \hat{E}_e^{(+)} (\vec{x} \times \vec{\nabla}) \hat{A}_e^{(+)} \delta_{\vec{k}, \vec{k}'}$$

$$= \sum_{\substack{\vec{k}, \vec{k}' \\ \lambda, \lambda'}} \frac{1}{2} \sqrt{\frac{\omega_k}{\omega_{k'}}} \hat{a}_{\vec{k}', \lambda'}^\dagger \hat{a}_{\vec{k}, \lambda} \vec{E}_{\vec{k}, \lambda}^* \cdot \vec{E}_{\vec{k}', \lambda'} + i\vec{\nabla}_{\vec{k}'} \delta_{(\vec{k}, \vec{k}')} \times \vec{k}$$

Though this is a finite sum, we are ultimately interested in the limit $V \rightarrow \infty$, when the sum goes to an integral and we can perform integration by parts.

$$\Rightarrow = \frac{1}{2} \sum_{\vec{k}, \lambda} \left(+i\vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda} \right)^\dagger \times \vec{k} \hat{a}_{\vec{k}, \lambda}$$

having used $\vec{E}_{\vec{k}, \lambda} \cdot \vec{E}_{\vec{k}', \lambda'} = \delta_{\lambda \lambda'}$

after integrating over $\vec{k}'$ with the delta fact.

Adding the conjugate term: $E_e^{(+)}(\vec{x} \times \vec{\nabla}) A_e^{(-)}$

$$\vec{J}_{\text{orbital}} = \frac{1}{2} \sum_{\vec{k}, \lambda} \left[(i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda})^\dagger \times \hbar \vec{k} \hat{a}_{\vec{k}, \lambda} + \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}^\dagger \times (i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda}) \right]$$

$$= \sum_{\vec{k}, \lambda} (i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda})^\dagger \times \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}$$

$$+ \underbrace{\sum_{\vec{k}, \lambda} \cancel{\hbar \vec{k} \hat{a}_{\vec{k}, \lambda}^\dagger} \times (i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda})}_{\text{Integration by parts} \Rightarrow 0}$$

Integration by parts $\Rightarrow 0$

$$\Rightarrow \vec{J}_{\text{orbital}} = \sum_{\vec{k}, \lambda} (i \vec{\nabla}_{\vec{k}} \hat{a}_{\vec{k}, \lambda})^\dagger \times \hbar \vec{k} \hat{a}_{\vec{k}, \lambda}$$

$$\text{or equivalent} = \sum_{\vec{k}, \vec{k}', \lambda} \hat{a}_{\vec{k}', \lambda}^\dagger \left[i \vec{\nabla}_{\vec{k}}, \delta(\vec{k} - \vec{k}') \times \hbar \vec{k} \right] \hat{a}_{\vec{k}, \lambda}$$

This is the "second quantized" form of orbital angular momentum. Recall from wave mechanics

$$\hat{L}_{\text{orbital}} = \hat{\vec{x}} \times \hat{\vec{p}} \doteq \left[-i \vec{\nabla}_{\vec{k}}, \delta(\vec{k} - \vec{k}') \right] \times \hbar \vec{k}$$

in momentum space? Thus, if we have an electromagnetic wave packet (beam/pulse) we generally carry both orbital and spin angular momentum.

Consider now the spin term

$$\int d^3x \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = -\frac{i\hbar}{2} \sum_{\vec{k}, \lambda, \lambda'} a_{\vec{k}, \lambda}^\dagger a_{\vec{k}', \lambda'} \vec{e}_{\vec{k}, \lambda}^* \times \vec{e}_{\vec{k}', \lambda'}$$

$$\underbrace{\int \frac{d^3x}{V} e^{i(\vec{k} - \vec{k}') \cdot \vec{x}}}_{\rightarrow \delta^{(3)}(\vec{k} - \vec{k}')} = -\frac{i\hbar}{2} \sum_{\vec{k}} \left[(\vec{e}_{\vec{k}, +}^* \times \vec{e}_{\vec{k}, +}) a_{\vec{k}, +}^\dagger a_{\vec{k}, +} + (\vec{e}_{\vec{k}, -}^* \times \vec{e}_{\vec{k}, -}) a_{\vec{k}, -}^\dagger a_{\vec{k}, -} \right]$$

As $\vec{e}_{\vec{k}, \pm} \equiv \frac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$

where $\vec{e}_1$ and $\vec{e}_2$ are two orthonormal vectors with $\vec{e}_1 \times \vec{e}_2 = \hat{k}$

$$\Rightarrow \vec{e}_{\vec{k}, \pm}^* \times \vec{e}_{\vec{k}, \pm} = \pm i \vec{e}_k$$

$$\Rightarrow \int d^3x \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}, +}^\dagger \hat{a}_{\vec{k}, +} - \hat{a}_{\vec{k}, -}^\dagger \hat{a}_{\vec{k}, -}) \vec{e}_k$$

Now $\int d^3x \frac{\vec{E}^{(+)} \times \vec{A}^{(-)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}, +} \hat{a}_{\vec{k}, +}^\dagger - \hat{a}_{\vec{k}, -} \hat{a}_{\vec{k}, -}^\dagger) \vec{e}_k$

$$= \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}, +}^\dagger \hat{a}_{\vec{k}, +} - \hat{a}_{\vec{k}, -}^\dagger \hat{a}_{\vec{k}, -}) \vec{e}_k \quad (\text{commutator cancels})$$

Finally note: $\vec{e}_{\vec{k}, \pm} \times \vec{e}_{\vec{k}, \pm} = 0$

$$\Rightarrow \int d^3x \vec{E}^{(+)} \times \vec{A}^{(+)} = \int d^3x \vec{E}^{(-)} \times \vec{A}^{(-)} = 0$$

Thus

$$\vec{J}_{\text{spin}} = \hbar \sum_{\vec{k}} (\hat{a}_{\vec{k},+}^\dagger \hat{a}_{\vec{k},+} - \hat{a}_{\vec{k},-}^\dagger \hat{a}_{\vec{k},-}) \vec{e}_{\vec{k}}$$

Each photon has intrinsic "spin" angular momentum. In the circularly polarized, plane wave basis, the photon has a definite helicity, ~~carries~~ carry one $\hbar$ of angular momentum along (opposite to) the direction of propagation $\vec{e}_{\vec{k}}$ for positive (negative) handed polarization.

The photon is spin $S=1$, yet there are only two states with definite projection of angular momentum, whereas, we might expect three ($2S+1 = 3$). This is a very subtle point coming from the fact the photon is massless. For more details see,

"Photons and Atoms",

(1d) Mapping photon spin onto a two-state Hilbert space

$$\text{Define } \hat{\vec{J}}_{\text{spin}} = \hat{J}_x \vec{e}_x + \hat{J}_y \vec{e}_y + \hat{J}_z \vec{e}_z$$

$$\text{where } \hat{J}_x = \frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+)$$

$$\hat{J}_y = \frac{\hbar}{2i} (\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+)$$

$$\hat{J}_z = \frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$$

This is the Schwinger representation of angular momentum connecting the "Boson algebra" $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ to the angular momentum algebra $[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k$

$$\text{Check: } [\hat{J}_x, \hat{J}_y] = \frac{\hbar^2}{4i} \left([\hat{a}_+^\dagger \hat{a}_-, -\hat{a}_-^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_-] \right)$$

$$= \frac{\hbar^2}{4i} \left[\underbrace{\hat{a}_+^\dagger \hat{a}_+}_{-1} ([\hat{a}_-^\dagger, \hat{a}_-]) - 2\hat{a}_-^\dagger \hat{a}_- \underbrace{([\hat{a}_+^\dagger, \hat{a}_+])}_{=-1} \right]$$

$$= i\hbar \left(\frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-) \right) = i\hbar \hat{J}_z \quad \checkmark$$

$$[\hat{J}_x, \hat{J}_z] = \frac{\hbar^2}{4} \left([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-] \right)$$

$$= \frac{\hbar^2}{4} (\hat{a}_+^\dagger \hat{a}_- (-1) - \hat{a}_+^\dagger \hat{a}_- (1) + \hat{a}_-^\dagger \hat{a}_+ (1) - \hat{a}_-^\dagger \hat{a}_+ (-1))$$

$$= -\frac{\hbar^2}{2} (\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+) = -i\hbar \hat{J}_y \quad \checkmark$$

$$\begin{aligned}
[\hat{J}_y, \hat{J}_z] &= \frac{\hbar^2}{4i} ([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-] \\
&\quad - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-]) \\
&= \frac{\hbar^2}{4i} (\hat{a}_+^\dagger \hat{a}_- (-1) - \hat{a}_+^\dagger \hat{a}_- (1) - \hat{a}_-^\dagger \hat{a}_+ (1) + \hat{a}_-^\dagger \hat{a}_+ (1)) \\
&= \frac{-\hbar^2}{4i} (\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+) \\
&= i\hbar \left[\frac{\hbar}{2} (\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+) \right] = i\hbar \hat{J}_x \quad \checkmark
\end{aligned}$$

The Schwinger representation is the "second quantized form" of the spin $1/2$ operators

$$\hat{J}_x = \frac{\hbar}{2} (|+\rangle_z \langle -| + |-\rangle_z \langle +|)$$

$$\hat{J}_y = \frac{\hbar}{2i} (|+\rangle_z \langle -| - |-\rangle_z \langle +|)$$

$$\hat{J}_z = \frac{\hbar}{2} (|+\rangle_z \langle +| - |-\rangle_z \langle -|)$$

"Second quantize" $| \pm \rangle_z \Rightarrow \hat{a}_\pm^\dagger$ create spin up or down

$\langle \pm |_z \Rightarrow \hat{a}_\pm$ annihilate spin up or down

Thus, we can easily map the spin angular momentum of the ~~photon~~ photon onto the Bloch sphere, also

known as the Poincaré sphere as we visited in PS#1