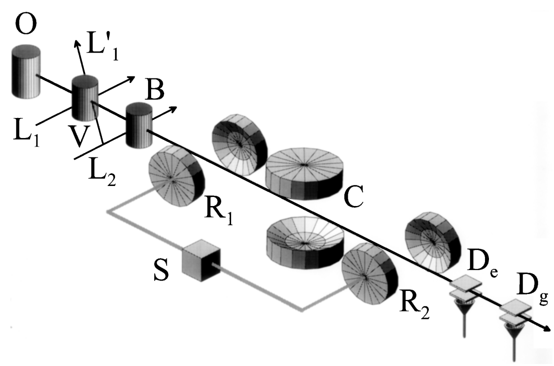
Physics 566: Quantum Optics I
Problem Set 10
Due Tueday November 25, 2025

Problem 1: Cavity QED (25 points)
In this problem we will explore some of Prof. Serge Haroche's seminal quantum optics experiments in cavity QED for which he was awarded the Nobel Prize in 2012. These demonstrate the quantum nature of the electromagnetic field – here microwave photons, as far from high energy photons as you can get!

Consider the following schematic:



Rubidium atoms, effusing from the oven *O*, are prepared at a well-defined time and selected with a well-defined speed in *V*. An atom can then be prepared in box *B* in one of two circular Rydberg states, $|g\rangle$ or $|e\rangle$, with principal quantum numbers *n*=50, 51, respectively, $\omega_{eg} = 51$ GHz. The atom passes through a high-Q superconducting cavity *C* such that one of the cavity modes, with frequency ω_c, is tuned near to the atomic transition $|e\rangle \leftrightarrow |g\rangle$. After passing through the cavity, an atom can be measured in detectors that determine if it is in state $|g\rangle$ or $|e\rangle$.

In addition, the quantum cavity *C* is sandwiched inside a *Ramsey Interferometer*. The Ramsey separated two zones, R_1 and R_2, correspond to classical microwave pulses that can apply $\pi/2$ -pulses on the $|e\rangle \leftrightarrow |g\rangle$ transition. In contrast to the standard atomic clock that we have studied, in which there is free evolution between the zones, here we have a quantum super-high-Q quantum cavity in the middle.

(a) Consider the case of a stream of atoms initially prepared in *B* in the excited state $|e\rangle$ and the cavity *C* in the vacuum. The cavity is tuned to resonance, $\omega_c = \omega_{eg}$ and the vacuum Rabi frequency is $2g_0/(2\pi) = 50$ kHz. The atoms are sent through the cavity and interact for a time *t* and then measured. The atomic beam flux is chosen very low, so that the separation time between atoms crossing the cavity, *T*, is much longer than the lifetime of a photon in *C*, so each atom sees a fresh vacuum. The Ramsey zones, R_1 and R_2, are not used in this experiment.

What is the probability of detecting the atom in the state $|g\rangle$ after it passes through *C* as a function of the interaction time. Sketch this and comment on your result.

(b) This apparatus can be used to detect the presence or absence of a single photon by looking at the correlation between two atoms that pass through the cavity. Consider the same operating conditions as part (a). The velocity is now chosen so that the interaction time is $2g_0 t = \pi$. A second atom is sent through the cavity in state $|g\rangle$ for the same interaction time.

Show the conditional probability of measuring the second atom in $|e\rangle$ conditioned on measuring the first in $|g\rangle$ is e^{-T/τ_c} where T is the time separation of the two atoms, and t_c is the cavity decay time. Comment on this result.

(c) Now let's employ the Ramsey cavities. One can use this to demonstrate the *transfer of quantum coherence* between two atoms, mediated by the quantum mode of the cavity. The operating conditions are again the same as above. With the quantum cavity C initially in the vacuum, the first Ramsey zone R_1 applies a $\pi/2$-pulse around x to the atom and prepares it in the superposition $\left(|e\rangle + i|g\rangle\right)/\sqrt{2}$. This atom passes through the quantum cavity C for an interaction time $2g_0 t = \pi$ and then measured to be in the state $|e\rangle$ or $|g\rangle$. After a time T, a second atom, initially in the state $|g\rangle$, is sent through the quantum cavity C for an interaction time $2g_0 t = \pi$. We apply here a $\pi/2$-pulse around x only on the second Ramsey zone R_2, with field phased-shifted by ϕ relative to the pulse in R_1. We read out the state of the two atoms.

Show the conditional probability of measuring the second atom in $|e\rangle$ conditioned on measuring the first in $|g\rangle$ is $(1+e^{-T/(2\tau_c)}\cos\phi)/2$. Give a Bloch sphere description of the transfer of coherence between the two atoms.

(d) A Ramsey interferometer can be used to measure the light-shift on a atom, as we have studied. Here we want to measure the *light shift of the quantized field* and show how this can be used to measure the absence or presence of a photon without destroying it (a so-called quantum nondemolition (QND) measurement). Suppose now that the cavity is slight *detuned* from resonance $\Delta = \omega_c - \omega_{eg} >> g_0$. The atoms are prepared in R_1 in $\left(|e\rangle + i|g\rangle\right)/\sqrt{2}$, and passed through the cavity with exactly n photons inside. The speed is sufficiently slow so that the initial "bare states" $|e,n\rangle$ and $|g,n\rangle$ adiabatically follow the "dressed states" of the coupled atom+cavity. The joint state after the interaction is $|n\rangle\left(e^{-i\delta_{e,n}}|e\rangle + ie^{-i\delta_{g,n}}|g\rangle\right)/\sqrt{2}$, where $\delta_{e,n}$ and $\delta_{g,n}$ are the phase shifts imparted to the states due to the light-shift (dressed) interaction. Note, the cavity still has exactly n photons – after the atom emerges, it neither absorbed or emitted a photons, but the quantized field caused a rotation of the atomic state in its Bloch sphere.

Find $\delta_{e,n}$ and $\delta_{g,n}$ and design the experiment to measure the photon number n.

Problem 2: Collapse a revival in the Jaynes-Cummings model (25 points)
One of the foundational results which demonstrated the quantum nature of the field was the study of Rabi oscillations of an atom in high-Q cavity (cavity QED).

Suppose at the initial time the atom is in the ground state by the cavity is in a *coherent state*: $|\Psi(0)\rangle_{AF} = |g\rangle \otimes |\alpha\rangle$. The joint atom-field state then evolves according to the Jaynes-Cummings Hamiltonian (we'll neglect here any loss or dissipation).

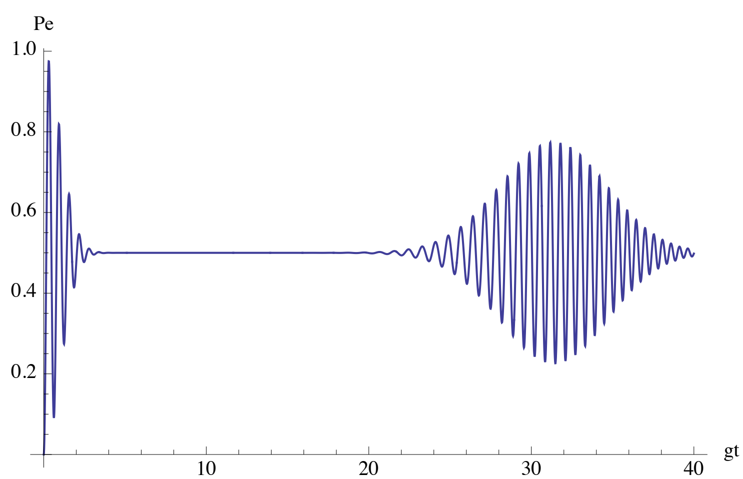
(a) Show that at a later time:

$$|\Psi(t)\rangle_{AF} = c_0|g\rangle\otimes|0\rangle + \sum_{n=1}^{\infty} c_n e^{-in\omega_0 t}\left(\cos\left(\sqrt{n}gt\right)|g\rangle\otimes|n\rangle + i\sin\left(\sqrt{n}gt\right)|e\rangle\otimes|n-1\rangle\right)$$

where $2g$ is the vacuum Rabi frequency and $c_n = \frac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}}$, and thus show that the probability to be in the excited state at time t, irrespective of photon number, is

$$P_e(t) = \sum_{n=1}^{\infty} \frac{\bar{n}^n e^{-\bar{n}}}{n!} \sin^2\left(\sqrt{n}gt\right), \text{ where } \bar{n} = |\alpha|^2 \ .$$

(b) Numerically calculate and plot P_e at a function of $0 \le gt \le 40$ for $\bar{n} = 25$. Your result should look as follows:



We see two distinctive features in this plot:

- The Rabi decay (collapse) after a few oscillations.
- After a long time they "revive" and the population starts oscillating again.

(c) The collapse is easily understood because we effectively have "inhomogeneous broadening." That is, we have different Rabi frequencies associated with different numbers of photons, $\Omega_n = \sqrt{n}\,2g$. We saw this kind of decay of Rabi oscillations early in the semester, when we had a *classical* distribution of intensities. The revival, by contrast, is a purely quantum effect of the field arising from the discrete frequency spectrum (Fourier sum).

Show for large $\bar{n}$, the expected (first) revival time, due to the discreteness of the photons is

$$t_{revive} \approx \frac{2\pi\sqrt{\bar{n}}}{g}$$. Compare with the plot in (b).

Note: The classical limit is of Rabi flopping in free space is intrinsically a multi-mode problem, and will exhibit these collapse and revivals.