Quantum state reconstruction

J. Grondalski
Department of Physics and Astronomy,
Albuquerque High Performance Computing Center,
University of New Mexico, Albuquerque, NM

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Abstract

We review the theory and experimental realizations of quantum state reconstruction techniques for systems with finite degrees of freedom and systems parameterized by continuous variables. The following systems in which quantum state reconstruction has been successfully achieved are reviewed: a spin S particle, the Wigner function of an atomic beam, the Q function of a vibrational mode of a molecule, the Wigner function of a mode of light, and the Wigner function of a motional state of a trapped ion.

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I. INTRODUCTION

Elementary quantum mechanics teaches that complete knowledge of a system is obtained when one knows the state function. Consider a state, $|\psi\rangle$, that is expanded in a complete basis, $\{|n\rangle\}$ (which we take to be discrete),

$$|\psi\rangle = \sum_n a_n |n\rangle,$$

where a_n are complex probability amplitudes such that $\sum |a_n|^2 = 1$. According to Max Born, a state vector manifests itself by associating, $|a_n|^2$, with the probability to measure a particular state $|n\rangle$. One can imagine testing this interpretation by going into the lab and preparing an ensemble of M identical systems and making a histogram of various outcomes. If state, $|n\rangle$, occurs N_n times, then N_n/M is an approximation of the probability of measuring the n^{th} state and it should be true that in the limit of large M,

$$\lim_{n\to\infty} \frac{N_n}{M} \to |a_n|^2.$$

This is a manifestly statistical concept because it implicitly assumes an ensemble of measurements are required for measuring this physical property. Contrast this to the measurement of the mass of a particle. An ideal mass spectrometer can uniquely specify a particle's mass in a single measurement. This deterministic notion of measurement is what is usually meant when one talks about "properties" of a particle. In what sense, statistical or deterministic, can one measure a particle's state vector? The answer to this question is that one can measure the state vector of a particle only in a statistical sense. This may seem counterintuitive because one commonly hears the phrase in a quantum lecture, '*...a particle is in a state,* $|\psi\rangle$*...*', which sounds like one is ascribing a quantum state to a particular particle, but if one derives one's notion of physical reality from what can actually be measured by experiment, then one must understand this statement as being fundamentally statistical, that is, derived from an ensemble of measurements on a collection of identical particles.

Experiments to directly measure the quantum state of a system have only been performed in the last decade. This is fairly surprising considering the fundamental position the concept of a state vector holds in quantum theory. For a general review of the field of quantum state measurement see Ref. [1, 2]. The quantum state reconstruction problem can be stated more generally as an experiment that reconstructs the density matrix of a system. Once the density matrix is known, all knowable information is obtained, but the density matrix is more general because it allows for the possibility of mixed states. Note, that when a system is in a pure state one can uniquely infer a state vector from a density matrix and vice versa, but for mixed states the state vector description is not unique. We will also refer to reconstructing quasiprobability distributions like the Glauber-Sudarshan P-function, Wigner function, and Q-function. These quasiprobability functions can be characterized by the Cahill-Glauber s-parameterized quasiprobability distribution [3], with $s = 1, 0, -1$ respectively. There is a one to one relationship between these quasiprobability functions and the density function, so whenever an experiment measures one of these quasiprobability distributions it may be inferred that the density matrix has been determined.

This article reviews all the experiments to date in which the quantum state has been successfully reconstructed in the lab. In Sec. II. A we start with a system associated with a small finite Hilbert space. We show how one can reconstruct an angular momentum density matrix from a series of Stern-Gerlach experiments. The next four systems are associated with infinite dimensional Hilbert spaces parameterized by a continuous variable (i.e. position and momentum). In Sec. II. B we show how one can tomographically reconstruct the Wigner function of an atomic beam. In Sec. II. C we show how one can reconstruct the quantum state of a vibrational mode of molecule excited by a short pulsed laser through endoscopic tomography. In Sec. II. D we look at the tomographic homodyne reconstruction of a quantum state of light. In Sec. II. E we show a novel non-tomographic technique to reconstruct both the density matrix in a Fock basis and the Wigner function of a trapped ion. Finally, in Sec. III we summarize our discussion.

II. QUANTUM STATE RECONSTRUCTION EXPERIMENTS

A Reconstruction of Spin Density Matrices

We first show how one can reconstruct a density matrix of a general spin S particle from series of Stern-Gerlach experiments oriented at different angles [4]. An experiment is currently being proposed to use this method to reconstruct the angular momentum density matrix of neutral atoms trapped in optical lattices [5]. We start with a density matrix which is an incoherent sum of n pure state projectors, $|\xi_n\rangle\langle\xi_n|$, weighted with a classical probability, P_n,

$$\rho = \sum_n P_n |\xi_n\rangle\langle\xi_n|.$$

This operator can be written in a basis of magnetic sub-levels associated with the spin, $\{|m\rangle\}$ $(m = -S, \cdots, S)$ for some arbitrarily chosen direction, $\hat{\mathbf{z}}$, where the, $\rho_{mm'}$ matrix element is given by,

$$\rho_{mm'} = \sum_n P_n \langle m|\xi_n\rangle\langle\xi_n|m'\rangle.$$

We next express the density matrix elements in terms of Stern-Gerlach measurements in various directions $\hat{\mathbf{n}}$. The projection of $\vec{\mathbf{S}}$ in the direction $\hat{\mathbf{n}}$ is related to $\hat{\mathbf{z}}$ by a unitary rotation $R(\hat{\mathbf{n}})$,

$$\vec{\mathbf{S}} \cdot \hat{\mathbf{n}} = R(\hat{\mathbf{n}}) S_z R^\dagger(\hat{\mathbf{n}}).$$

We can write the magnetic eigenstates in the direction of $\vec{\mathbf{S}} \cdot \hat{\mathbf{n}}$ in the $\hat{\mathbf{z}}$-basis,

$$|m\rangle_{\hat{\mathbf{n}}} = R(\hat{\mathbf{n}})|m\rangle_{\hat{\mathbf{z}}} = \sum_{m'=-S}^{S} R_{m'm}(\hat{\mathbf{n}})|m'\rangle_{\hat{\mathbf{z}}},$$

where $R_{m'm}(\hat{\mathbf{n}})$ is the well-known rotation matrix,

$$R_{m'm}(\hat{\mathbf{n}}) = {}_{\hat{\mathbf{z}}}\langle m'|R(\hat{\mathbf{n}})|m\rangle_{\hat{\mathbf{z}}} = e^{-i(m-m')\phi}\, d^{(s)}_{m',m}(\theta),$$

where $\{\phi, \theta\}$ are the usual polar angles and $d^{(s)}_{m',m}(\theta)$ is the Wigner d function defined in Ref. [6]. Now consider the projection of the μ^{th} magnetic sublevel in the direction $\hat{\mathbf{n}}$ direction written in a $\hat{\mathbf{z}}$ basis,

$$|\mu\rangle_{\hat{\mathbf{n}}\hat{\mathbf{n}}}\langle\mu| = \sum_{m'=-S}^{S}\sum_{m=-S}^{S} R_{m'\mu}(\hat{\mathbf{n}})R^*_{m\mu}(\hat{\mathbf{n}})|m'\rangle_{\hat{\mathbf{z}}\hat{\mathbf{z}}}\langle m|.$$

If one now defines the probability for a particle to be in state μ, to be the trace of this projector,

$$p_\mu = \mathrm{Tr}[\hat{\rho}\;|\mu\rangle_{\hat{\mathbf{n}}\hat{\mathbf{n}}}\langle\mu|],$$

inserting previous result into the expression above gives,

$$p_\mu = \sum_{m'=-S}^{S}\sum_{m=-S}^{S} e^{-i(m-m')\phi}\; d^{(s)}_{m',\mu}(\theta)d^{(s)}_{m,\mu}(\theta)\;\rho_{m,m'}.$$

The coefficients in front of $\rho_{m,m'}$ are known. If we measure the populations of the $(2S+1)$ magnetic sublevels at $(4S+1)$ different angles ϕ, then we see that we have $(2S+1)(4S+1)$ equations. Since there are $4S(S+1)$ independent unknown matrix elements (note: not all of them are independent because of hermiticity and the trace condition), the problem is isomorphic to the mathematical problem of solving a set of overdetermined algebraic equations which can in general be solved numerically using a singular value decomposition [7]. Notice that larger systems require more Stern-Gerlach measurements and the number of measurements scales like $\sim S^2$. This is associated with the fact that the number of independent density matrix elements scales as the square of the dimension of the Hilbert space. The number of measurements can be reduced drastically if additional information is known about the system. For instance, in the case when one knows that the state is a pure state, the density matrix can be reconstructed with just three Stern-Gerlach measurements [8]. A similar but not identical method to the one described above has been used to measure the angular density matrix of a beam of collisionally produced hydrogen. The density matrix elements were obtained from multipole measurements (electronic charge density, current density) and the use of sophisticated fitting algorithms [9].

B Reconstruction of Atomic Beams

We now turn to tomographic reconstruction techniques used to measure the quantum mechanical state of the center of mass of a particle. The word *'tomography'* derives from the method's similarity to the well known medical imaging technique in which a 3-dimensional internal images of the human body can be reconstructed from a series of 2-dimensional images taken at different angles around the body. We start with atomic beam experiments because they illustrate most simply the tomographic technique in the context of quantum state reconstruction. Consider a two dimensional atomic beam whose direction of propagation is in the $\hat{\mathbf{z}}$ direction and whose transverse component is in the $\hat{\mathbf{x}}$ direction. Suppose we wish to reconstruct the transverse beam profile $\psi(x)$ at some initial time $t = 0$. Measuring $|\psi(x)|^2$ at this time does not allow one to obtain $\psi(x)$ because phase information is lost by taking the modulus square. Suppose instead one measures $|\psi(x,t)|^2$ of the beam at time, t, down stream. One can think of this as really measuring the probability, $|\psi(\xi,t)|^2$, of a particle to be at a coordinate, ξ, that is a combination of position and momentum,

$$\xi = x + (\frac{t}{M})p_x,$$

where p_x is the momentum in the $\hat{\mathbf{x}}$ direction and M is the mass of the particle. One can view this new coordinate as a condition on the combination of initial position and momentum that confines a particle to be at coordinate, ξ, at time, t. A measurement of the atomic density at this time is a measurement of the probability distribution along a slice of phase space (see Fig. 1). The Wigner function, $W(x,p)$, is a quasiprobability which can be used to reproduce the classical probability distribution when the distribution is integrated or projected onto slices of phase space,

$$|\psi(\xi,t)|^2 = \int_{C(\xi)} W(x,p)dxdp_x.$$

By repeating this measurement at different times, one can measure the probability distribution for many different slices through phase space. This is the reason for the name

'Tomography'. Once the probability distributions of the various slices has been obtained the data can be inverted uniquely to obtain the Wigner distribution. The transformation that accomplishes this transformation is called a Radon transform. Although the details of the Radon transform are beyond the scope of this paper, it can be thought of an integral transform, like a Fourier transform, which is associated with a certain kernal [10].

Partial reconstruction of a Wigner function of an atomic beam profile of Helium atoms after impinging on two closely spaced slits from the atomic counts down stream has been carried out experimentally by Kurtsiefer et. al. [11]. After passing through the double slit, the atoms are in a superposition of two macroscopically spaced positions ($\sim 8\mu$m) which is a highly non-classical state of matter. The Wigner function, therefore, should have many regions of negativity which are indicators of non-classical motion. The experimentally reconstructed Wigner function does indeed have negative regions that are in fair agreement with the theoretical predictions (See Fig. 2). The discrepancy can be attributed to the fact that time is always positive, so one can only measure slices of phase space with positive slopes. Thus, the integrated probability data from a large region of phase space is inaccessible causing errors in the the Radon transform. This can be partially overcome with sophisticated algorithms that optimize the partial information for the Radon transform, but for the reconstruction to be complete, atom optic elements that produce effects equivalent to time reversal (like phase conjugate mirror in optics) must be developed.

C Reconstruction of a Vibrational Mode of a Molecule

We next look at a novel technique that uses the "magic" of a harmonic oscillator to reconstruct the Q-function of a vibrational mode of a molecule. Consider a molecule that is excited by a short pulse laser from the ground state into an excited state and whose potential can be approximated as parabolic. If the excitation process is fast compared to the vibrational dynamics, then the system can be viewed as a molecular wave packet with an initial state which proceeds in time according to a harmonic oscillator hamiltonian. (See

Fig. 3). A classical harmonic oscillator with initial position q_o and initial momentum p_o will evolve dynamically according to,

$$q(t) = q_o \cos(\omega t) + (p_o/\omega m) \sin(\omega t)$$
$$p(t) = -q_o \sin(\omega t) + (p_o/\omega m) \cos(\omega t).$$

A particle's position, $|\psi(x)|^2$ after a quarter of a period ($\tau = 2\pi/\omega$) is a measure of it's momentum distribution, $|\psi(p_x)|^2$. In general, the quasiprobability distribution of the particle is parameterized in such a way that $|\psi(x)|^2$, for every time during one period of oscillation, is associated with a slice of phase space of the original wave function (See Fig. 4). Thus, one simply needs to monitor the position of the wave packet as a function of time over one period. The wave packet can be monitored with a short pulse laser by measuring the molecular spectrum as a function of time according to ideas developed by Eberly and Wodkiewicz [12]. Fig. 3 shows that one can map a particle's position onto it's spectrum because each position in the potential is associated with a unique energy. It can be shown that the time dependent molecular spectrum, $S(\Omega, t)$, is related to the Q function or Husimi distribution of the vibrational mode of the molecule.

$$S(\Omega, t) = \int dp\, Q(p(t), q(t)),$$

where Ω is the frequency of light in the time dependent molecular spectrum and t is taken over one period. The condition that the potential be harmonic can be relaxed slightly as long as the anharmonicities do not cause the wave packet to change shape during one period. If the wave packet meets this condition then the wave packet's Q function can be dynamically monitored in real time. This experiment was carried out with sodium dimers by Dunn et. al. [13]. They were able to reconstruct a Q function (See Fig. 5) that had some blurring due to poor resolution caused by probing a small energy spectrum with a large band width (short pulse) light source. However, this technique could be used as a diagnostic for quantum control of molecular wave packets which is important, for example, in chemical reaction control.

D Reconstruction of Quantum States of Light

In this section we discuss the first experimental reconstruction of a quantum state with a continuous variable, the homodyne reconstruction of a quantum state of light, which was first proposed by Vogel et. al. [14] and performed by Smithey et. al. [15]. In this system the two quadrature components of a light field, $\{p(0), p(\pi/2)\}$, play the role of position and momentum. To see how this measurement scheme works consider that an oscillating electric field can in general be written as,

$$E(t) = E_o(p(0)\cos(\omega t + \theta) + p(\pi/2)\sin(\omega t + \theta)).$$

The phase, θ, between the quadratures can be associated with a slice through phase space at this angle (See Fig. 4). The quadrature of a light field can be measured with homodyne detection. Homodyne detection consists of measuring the intensity of a signal interfering with strong, known field called the local oscillator,

$$E_{lo}(t) = E_{lo}\cos(\omega t + \theta).$$

Note that with the local oscillator the experimenter has control of the phase. The set of measurements corresponding to slices in phase space are just homodyne measurement at different phase shifts of the local oscillator and the Wigner function can be reconstructed as before. A coherent state with a small mean number of photons ($\bar{n} \sim 1.2$) was reconstructed according to the above method [15] (See Fig. 6). Experiments that reconstruct the Wigner function of a state of light are interesting for their insight into the nature of quantum mechanics, but from a practical point of view once one knows the Wigner function, the average and higher moments of any observable can be computed. This is potentially useful because it allows one to "measure" photo-count statistics efficiently without having to use low flux light sources [16].

E Reconstruction of Motional States of Trapped Ions

The last quantum state reconstruction scheme we discuss is the reconstruction of the motional state of a trapped ion [17, 18]. This scheme is interesting because it uses a non-tomographic method to reconstruct a quantum state with a continuous variable. The system consists of an ion trapped in an rf Paul trap. The system is schematically depicted in Fig. 7. An ion is initially in an internal state $|1\rangle$ whose motional states are well described by an harmonic oscillator, $|n\rangle$. The initial state is then weakly coupled to another internal state, $|2\rangle$, with two laser beams that are red and blue detuned by one unit of oscillator energy, $\hbar\omega$. This causes Raman transitions between $(|1,0\rangle \rightarrow |2,1\rangle, |1,1\rangle \rightarrow |2,2\rangle, \cdots |1,n\rangle \rightarrow |2,n+1\rangle)$ whose Rabi oscillation frequencies, $(\Omega_{01}, \Omega_{12}, \cdots, \Omega_{nn+1})$, scale as $\sqrt{n}$. If we neglect higher order couplings we see that the problem is equivalent to a Jaynes-Cummings model in which a series of two level atoms are coupled by harmonic oscillators with different frequencies. Internal state $|2\rangle$ is then strongly coupled to another level $|3\rangle$ whose florescence can be used to probe the system. One can also cause a coherent displacement, α, of the well associated with level $|1\rangle$ by simply displacing the well "suddenly" compared to a unit of oscillator energy. It can then be shown that for a given coherent displacement, α, the probability that that the ion is in state $|2\rangle$ at some time, t, is given by

$$P_2(t,\alpha) = \frac{1}{2}(1 + \sum_{n=0}^{\infty} Q_n(\alpha)\cos(2\Omega_{n,n+1}t)e^{-\gamma}),$$

where γ is a decoherence factor and $Q_n(\alpha)$ are the populations of the motional eigenstates. The transition rate between states $|1\rangle \rightarrow |2\rangle$ is fast compared to a unit of oscillation energy so the motion of the ion can be probed essentially instantaneously. The density matrix in a Fock basis can now be reconstructed in the same way as the angular momentum density matrix was reconstructed in Sec. II. A. To see this substitute the unitary rotation matrix with the unitary displacement matrix,

$$R(\theta) \rightarrow D(\alpha)$$

$$\text{where } D(\alpha) = e^{\alpha\dagger\hat{a} - \alpha^*\hat{a}}.$$

The density matrix elements, in a Fock basis, of an ion can now be found like before with a singular value decomposition. Furthermore, once the $Q_n(\alpha)$ terms are known through the above method, one can then compute the Wigner function at the phase space point α [19] through,

$$W(\alpha) = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n Q_n(\alpha).$$

This relation is very useful because it allows one to measure the Wigner function directly at α without having to perform the global measurements on the system associated with tomography. Conceptually this result follows from the fact that the Wigner function can be measured directly at the origin of phase space and that one can measure other points in phase space through unitary displacements. A similar non-tomographic approach was proposed by Davidovich et. al. Ref. [20] for the measurement vibrational modes of a molecule. In Fig. 8 we show the reconstructed Wigner function of a $^9\mathrm{Be}^+$ ion in a highly non-classical motional state associated with a $n = 1$ Fock state [17].

III. CONCLUSIONS

We summarize by noting that these quantum state reconstruction experiments fall into three main categories. The first reconstruction technique requires one to associate the density operator with a set of measurable observables which can be used to invert a set of over complete equations via a singular value decomposition. Examples of this technique include the reconstruction of an angular momentum density matrix of collisionally produced hydrogen and the density operator of a motional state of a trapped ion. The second technique inverts a set of classical probability distributions associated with slices of phase space with a Radon transform to reconstruct a state vector. Examples of this were the reconstruction of quasiprobability distributions associated with the quantum states of an atomic beam, a vibrational mode of a molecule, and a coherent state of light. The last technique uses coherent displacements of the origin of phase space, where the Wigner function can be computed directly, to compute the Wigner function at different points in phase space. The ion trap experiment is an example of this last technique. One notices that all these systems require an ensemble of measurements even though, strictly speaking, the quantity being measured is not a probability. In fact it has been shown that it is impossible to determine the quantum state of a single system by a succession of measurements on the same system because each successive measurement uncontrollably perturbs the quantum state [21]. This shows that the concept of a state vector can only be understood in a statistical sense. These experiments are interesting because they give insight, through concrete experiments, into the interpretation of a quantum state.

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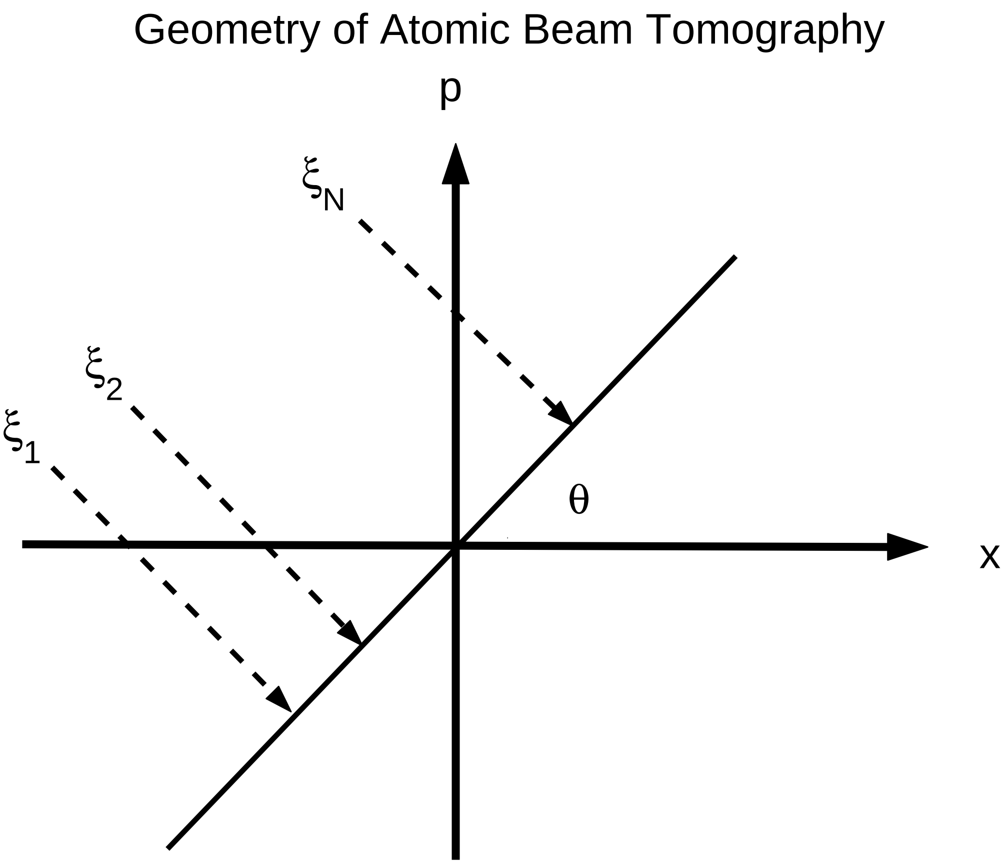

FIG. 1: $\xi_1, \xi_2, \cdots, \xi_N$ are lines of integration along a quasiprobability distribution which determines a positive probability distribution along a slice of phase space determined by θ.

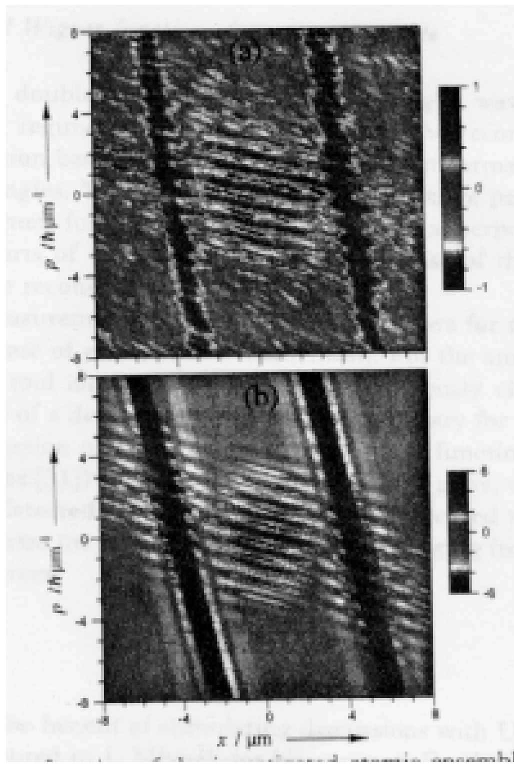

FIG. 2: Experimental results of a partial reconstruction of a Wigner function for free atoms after passing through a pair of closely spaced slits. (From Ref. [11])

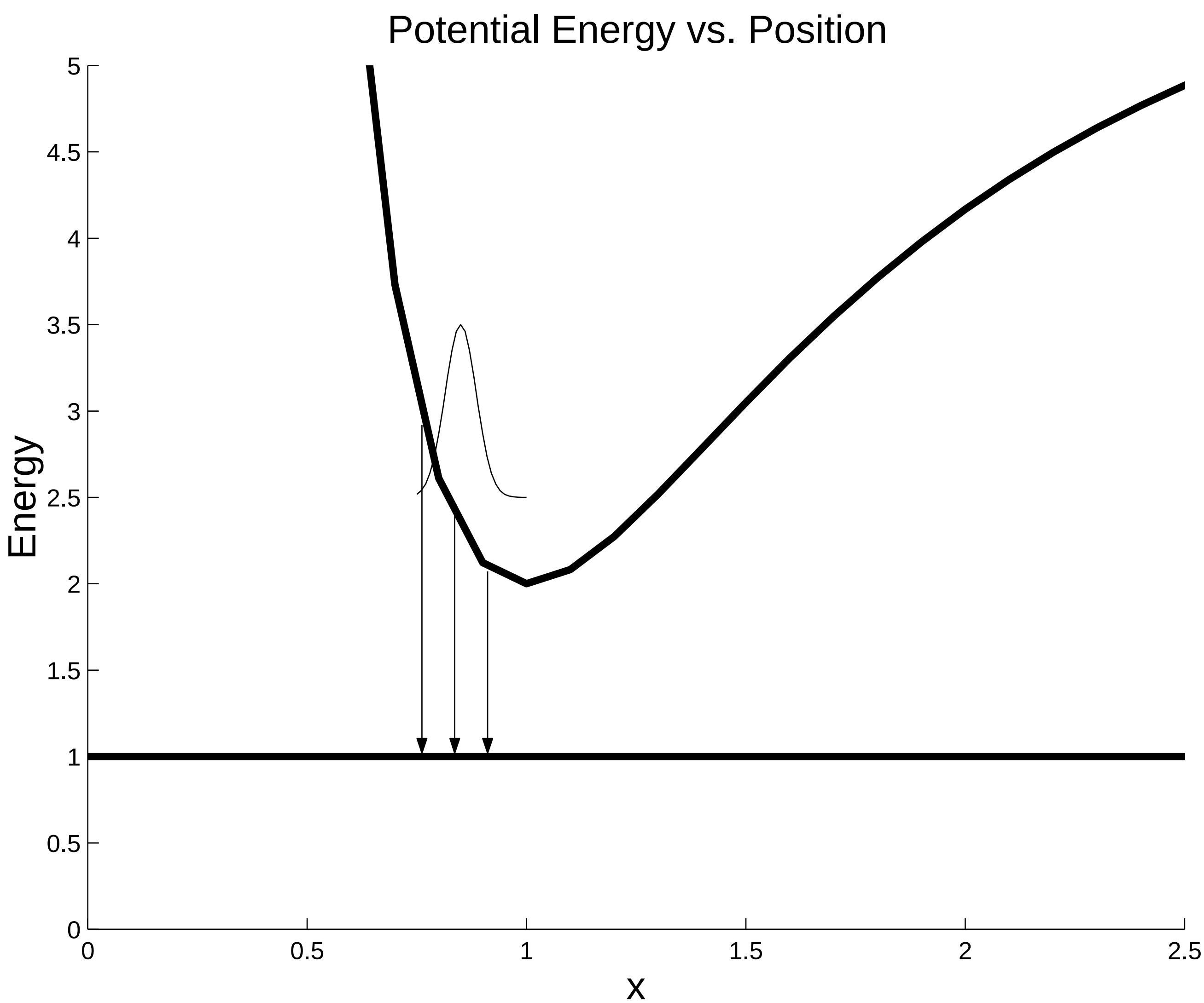

FIG. 3: Schematic that shows how the spectrum of an excited wave packet is a map of the position of the wave packet.

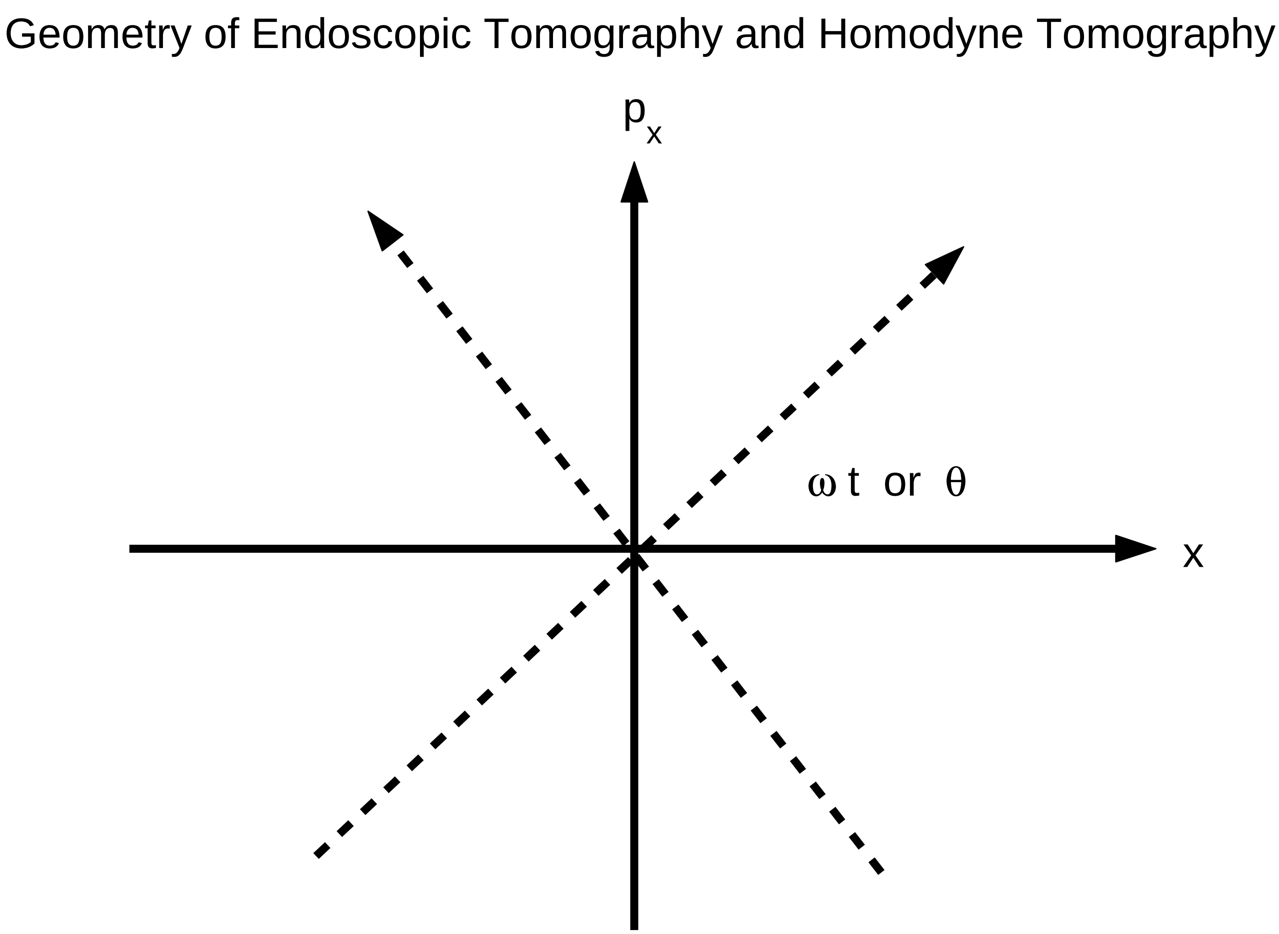

FIG. 4: For endoscopic tomography the harmonic oscillator causes the quasiprobability distribution to be rotated in phase space by ωt. For homodyne tomography the phase of the local oscillator allows one to rotate the phase space by θ.

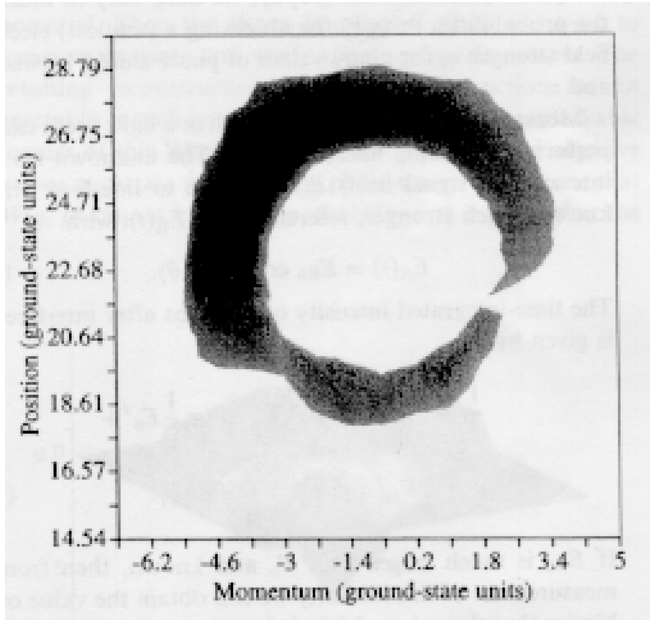

FIG. 5: The result of an experimental reconstruction of a Q function representing the quantum state of a molecular oscillator. (From Ref. [13])

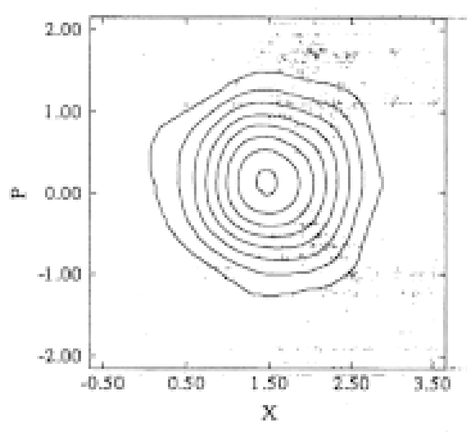

FIG. 6: Measured Wigner function of a coherent state light with an average number of photons, $\bar{n} \sim 1.2$. (From Ref. [15])

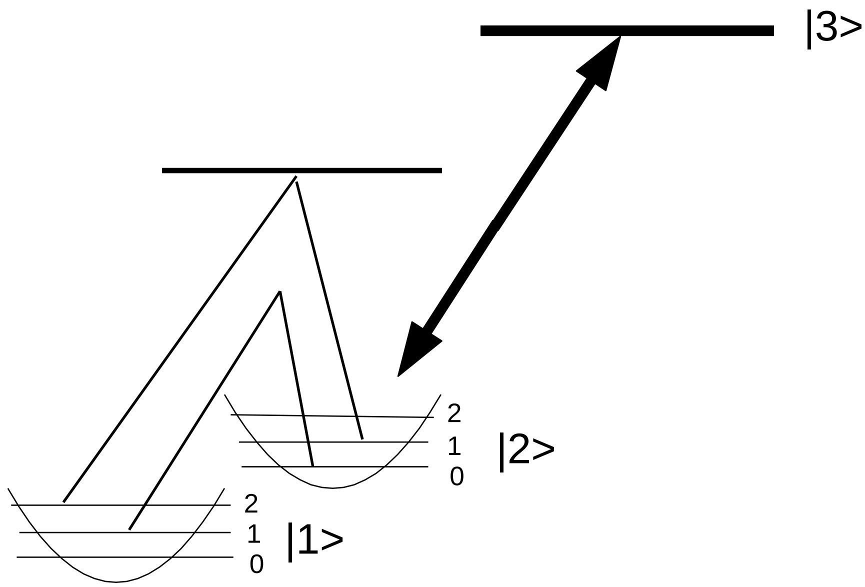

FIG. 7: Schematic of the experiment to reconstruct the motional states of a trapped ion. The motional state $|1, n\rangle$ is coupled to the motional state $|2, n+1\rangle$ with a Rabi frequency $\Omega_{n,n+1}$, through an appropriate set of Raman beams.

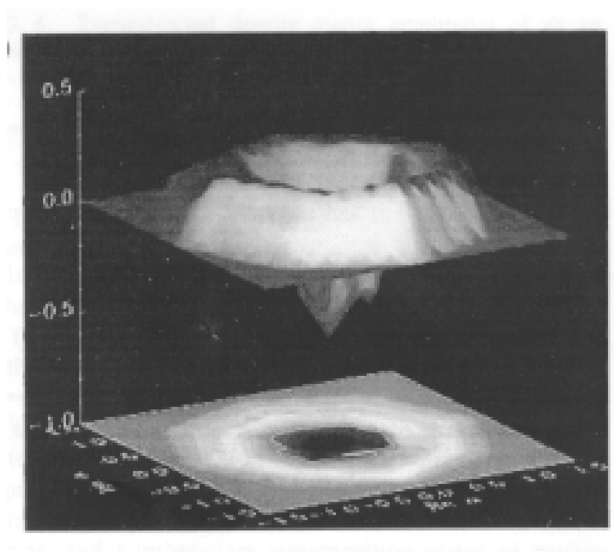

FIG. 8: Experimentally reconstructed Wigner function for an ion in a motional state associated with a $n = 1$ Fock state. (From Ref. [17])