Physics 566, Spring 2002

Lecture 2: Two-level systems

One of the most fundamental systems in quantum mechanics is the simplest — a two-level system, i.e. something describable by a Hilbert space of dimension two. Examples include spin 1/2 and an atom where only two orthogonal eigenstates are important. Since Hilbert spaces of the same dimension are isomorphic, all such systems share the same description.

Spin 1/2 particles have an intrinsic angular momentum. Thus all two level systems have an intimate relation to the theory of angular momentum in quantum mechanics (the group SU(2)) and magnetic resonance.

SU(2) for spin-1/2

Angular momentum $\hat{\vec{S}}$

$$[\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{S}_k$$

$$\hat{\vec{S}}^2 = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\hat{\mathbb{1}}$$

Standard basis: Eigenstates of $\hat{\vec{S}}^2$ and $\hat{S}_z$

$$\hat{S}_z |\pm_z\rangle = \pm\tfrac{1}{2} |\pm_z\rangle$$

Raising and lowering operators

$$\hat{S}_\pm = \hat{S}_x \pm i \hat{S}_y \quad \text{("spherical basis")}$$

$$\hat{S}_+ |+_z\rangle = \text{null} \qquad \hat{S}_- |+_z\rangle = |-_z\rangle$$

$$\hat{S}_+ |-_z\rangle = |+_z\rangle \qquad \hat{S}_- |-_z\rangle = \text{null}$$

$$[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm \qquad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$$

Pauli-spin operators / matricies

Define $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$ (Note: Define $\hat{S}_\pm = \hbar\hat{\sigma}_\pm$ factor $2P_0$)

$\sigma_z|\pm\rangle_z = \pm 1\ |\pm\rangle_z$ $\qquad \hat{\sigma}_\pm = \frac{\hat{\sigma}_x \pm i\hat{\sigma}_y}{2}$

$\sigma_+|+\rangle_z = \text{null}$, $\sigma_+|-\rangle_z = |+\rangle_z$, $\sigma_-|+\rangle_z = |-\rangle_z$, $\sigma_-|-\rangle_z = \phi$

$\hat{\sigma}_z = |+\rangle_z\langle+| - |-\rangle_z\langle-|$ $\qquad \hat{\sigma}_\pm = |\pm\rangle_z\langle\mp|$

$\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$, $\qquad \hat{\sigma}_y = \frac{\hat{\sigma}_+ - \hat{\sigma}_-}{i}$

In basis $\{|+\rangle_z, |-\rangle_z\}$ the matrix representation of the $\hat{\sigma}_i$ operators are known as the Pauli spin matricies

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad (\text{rank } 1)$$

Properties

$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$ $\qquad \{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}$ (anticommute)

$[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$ $\qquad [\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$

$\hat{\sigma}_i^2 = \hat{\mathbb{1}} \Rightarrow$ Since Hermitian, σ_i are Unitary

$Tr(\hat{\sigma}_i) = 0$ $\qquad$ $Det(\hat{\sigma}_i) = -1$

$= \sum$ eigenvalues $\qquad$ $= \Pi$ eigenvalues

$\hat{\sigma}_i\hat{\sigma}_j = i\epsilon_{ijk}\hat{\sigma}_k + \delta_{ij}\hat{\mathbb{1}}$

The spin-1/2 Pauli matrices plus identity form a "complete orthogonal basis" of 2×2 matrices with trace inner product.

Aside:
- $n \times n$ matrices are themselves a n^2-dim vector space
- Inner product $\mathrm{Tr}(\hat{A}^\dagger \hat{B}) = (\hat{A}|\hat{B})$
- E.g. the operators $\{|n\rangle\langle m| \equiv \hat{T}_{nm}\}$ form an orthonormal basis of operators

$$\mathrm{Tr}(\hat{T}_{nm}^\dagger \hat{A}) = \langle n|\hat{A}|m\rangle \quad \text{matrix element}$$

$$\mathrm{Tr}(\hat{\sigma}_i \hat{\sigma}_j) = 2\delta_{ij} \quad \text{where } \hat{\sigma}_0 \equiv \hat{\mathbb{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Given any 2×2 matrix (operator)

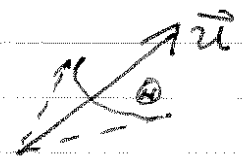
$$\hat{A} = \frac{1}{2}\sum_{i=0}^{1} a_i \hat{\sigma}_i = \frac{1}{2}(a_0 \hat{\mathbb{1}} + \vec{a}\cdot\vec{\hat{\sigma}})$$

where $a_0 = \mathrm{Tr}(\hat{A}\hat{\sigma}_0) = \mathrm{Tr}(\hat{A})$

$i = 1,2,3 \quad a_i = \mathrm{Tr}(\hat{A}\hat{\sigma}_i)$

Rotations

$SU(2)$ $\hat{D}(\Theta, \vec{u})$



axis-angle parameters

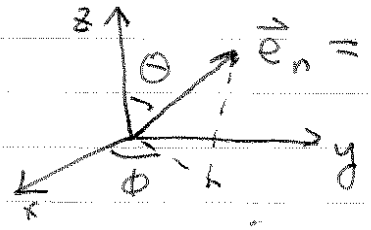
$$\hat{D}(\Theta,\vec{u}) = e^{-\frac{i}{\hbar}\Theta\,\vec{u}\cdot\vec{S}} = e^{-\frac{i}{2}\Theta\,\vec{u}\cdot\vec{\hat{\sigma}}}$$

$$= \cos\frac{\Theta}{2}\hat{\mathbb{1}} - i\sin\frac{\Theta}{2}\,\vec{u}\cdot\vec{\hat{\sigma}}$$

(familiar form)

Spin along an arbitrary direction

Consider unit vector along direction θ, ϕ



$$\vec{e}_n = \sin\theta(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y) + \cos\theta\,\vec{e}_z$$

Define spin along $\vec{e}_n$: $\hat{\sigma}_n \equiv \vec{e}_n \cdot \vec{\sigma}$

$$\hat{\sigma}_n = \left(\hat{D}_z(\phi)\hat{D}_y(\theta)\right)\hat{\sigma}_z\left(\hat{D}_z(\phi)\hat{D}_y(\theta)\right)^{\dagger} \quad \text{(Passive)}$$

$\Rightarrow$ Spin-up along $\vec{e}_n$ $\quad \hat{\sigma}_n|+_n\rangle = +1\,|+_n\rangle$

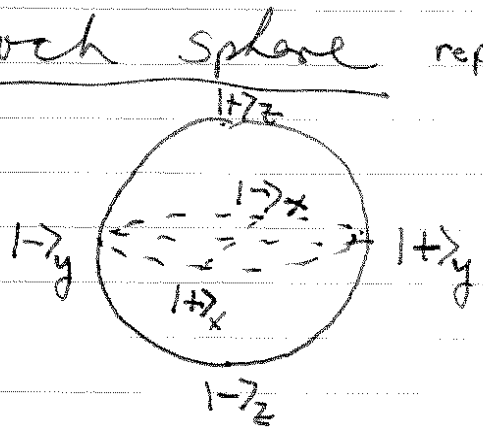
$$|+_n\rangle = \hat{D}_z(\phi)\hat{D}_y(\theta) = e^{-i\frac{\phi}{2}}\cos\tfrac{\theta}{2}|+_z\rangle + e^{i\frac{\phi}{2}}\sin\tfrac{\theta}{2}|-_z\rangle$$

$$\equiv \cos\tfrac{\theta}{2}|+_z\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|-_z\rangle$$

Fact: Every state in 2D Hilbert space is $|+_n\rangle$ for some $\vec{e}_n$ (see Prob. set)

$\Rightarrow$ Every (pure) state of two-level system is in 1-to-1 correspondence with points on the surface of a sphere of unit radius

Bloch sphere representation of $|\psi\rangle = c_+|+_z\rangle + c_-|-_z\rangle$



Antipodal points are orthogonal

"Bloch vector"

$$\vec{e}_n = \langle\psi|\vec{\sigma}|\psi\rangle$$

Sometimes

$$u \equiv \langle\hat{\sigma}_x\rangle = 2\,\mathrm{Re}(c_+ c_-^*)$$

$$v \equiv \langle\hat{\sigma}_y\rangle = 2\,\mathrm{Im}(c_+ c_-^*)$$

$$w \equiv \langle\hat{\sigma}_z\rangle = |c_+|^2 - |c_-|^2$$

Mixed states: the "Bloch Ball"

Consider a general density operator $\hat{\rho}$ for a two-level system. Like any operator on this space it can be written

$$\hat{\rho} = \frac{1}{2}\left(\underbrace{\mathrm{Tr}(\hat{\rho})}_{=1}\hat{\mathbb{1}} + \vec{Q}\cdot\vec{\hat{\sigma}}\right)$$

where $\vec{Q} = \mathrm{Tr}(\hat{\rho}\,\vec{\sigma}) = \langle\vec{\sigma}\rangle \equiv$ Bloch vector (real)

$\Rightarrow$ Any state for a two-level system is in 1-to-1 correspondence with a Bloch vector

Components:

$$\begin{cases} u = Q_x = \mathrm{Tr}(\hat{\rho}\hat{\sigma}_x) = \rho_{-+} + \rho_{+-} = 2\,\mathrm{Re}(\rho_{-+}) \\ v = Q_y = \mathrm{Tr}(\hat{\rho}\hat{\sigma}_y) = \dfrac{\rho_{-+} - \rho_{+-}}{i} = 2\,\mathrm{Im}(\rho_{-+}) \\ w = Q_z = \mathrm{Tr}(\hat{\rho}\hat{\sigma}_z) = \rho_{++} - \rho_{--} \quad \text{(Population inversion)} \end{cases}$$

Pure vs. mixed state

$$\hat{\rho}^2 = \left[\frac{1}{2}\left(\hat{\mathbb{1}} + \vec{Q}\cdot\vec{\hat{\sigma}}\right)\right]^2 = \frac{1}{4}\left(\hat{\mathbb{1}} + Q_i Q_j\,\hat{\sigma}_i\hat{\sigma}_j + 2\vec{Q}\cdot\vec{\hat{\sigma}}\right)$$

$$= \frac{1}{4}\left(\hat{\mathbb{1}} + |\vec{Q}|^2\hat{\mathbb{1}} + 2\vec{Q}\cdot\vec{\hat{\sigma}}\right)$$

$$\mathrm{Tr}(\hat{\rho}^2) = \frac{1+|\vec{Q}|^2}{4}\underbrace{\mathrm{Tr}(\hat{\mathbb{1}})}_{=2} + 2Q_i\,\underbrace{\mathrm{Tr}(\hat{\sigma}_i)}_{\to 0}$$

$$\Rightarrow \boxed{\mathrm{Tr}(\hat{\rho}^2) = \frac{1}{2}\left(1 + |\vec{Q}|^2\right)}$$

$\Rightarrow$

Pure state	$\lvert\vec{Q}\rvert^2 = 1$	unit vector	(sphere)
Mixed state	$\lvert\vec{Q}\rvert^2 < 1$	$\lvert\vec{Q}\rvert = 0$	completely mixed