Phys. 566 : Quantum Optics

<u>Lecture 3:</u> Magnetic Resonance and Rabi flopping

All coherent laser spectroscopy based on magnetic resonance

I. Rabi 1939: $\begin{cases}\text{measured g-factor of hyperfine structure} \\ \text{Lamb shift}\end{cases}$

Problem of manipulating spin

Hamiltonian, like any operator on 2D space

$$\hat{H} = \underbrace{A\hat{\mathbb{1}}}_{\text{constant (zero of energy)}} + \vec{B}\cdot\vec{\sigma}$$

$\Rightarrow$ All Hamiltonians are equivalent to spin in magnetic field.

— Static $\vec{B}$ field $\quad \hat{H}_0 = -\hat{\vec{\mu}}\cdot\vec{B}$

$$\hat{\vec{\mu}} = \gamma\hat{\vec{S}} = \frac{\hbar\gamma}{2}\vec{\sigma} \qquad \gamma = \text{gyromagnetic ratio} = \pm g\mu_0$$

$$\Rightarrow \quad \hat{H}_0 = -\frac{\hbar}{2}\gamma\vec{B}\cdot\vec{\sigma} = -\frac{\hbar\Omega_L}{2}\vec{e}_n\cdot\vec{\sigma}$$

$$\gamma\vec{B} = \Omega_L\,\vec{e}_n \qquad \Omega_L = \gamma|\vec{B}| \quad \text{Larmor frequency}$$

Eigenspectrum $\quad E_{\pm} = \mp\frac{\hbar\Omega_L}{2} \;\begin{matrix}\nearrow |+_n\rangle \\ \searrow |-_n\rangle\end{matrix}$

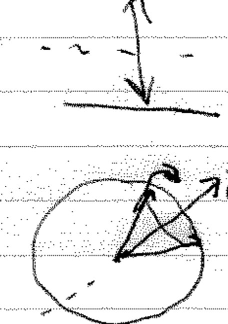
Zeeman effect

$|-_n\rangle$

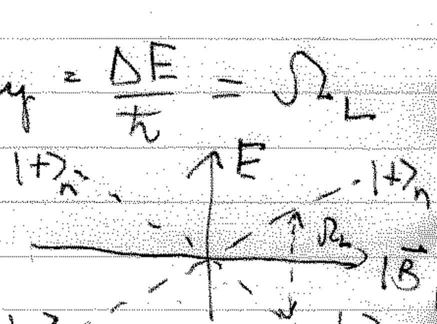
ω_0 = Bohr frequency $= \frac{\Delta E}{\hbar} = \Omega_L$

$|+_n\rangle$

Precession



spin



Magnetic Resonance

• Apply static $\vec{B}$ field (call it $-\vec{e}_z$ direction)

$$\vec{B}_0 = -B_{\parallel}\vec{e}_z \quad \text{↑"longitudinal"}$$

$$\hat{H}_0 = -\hat{\vec{\mu}}\cdot\vec{B}_0 = \hbar\frac{\omega_0}{2}\hat{\sigma}_z$$

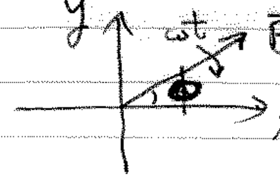
Energy splitting $\Delta E = \hbar\omega_0 = \hbar(\gamma B_{\parallel})$

• Drive system resonantly between $|\pm_z\rangle$ by applying an oscillating field at ω

$$\hat{H}_1(t) = -\hat{\vec{\mu}}\cdot\vec{B}_1(t), \text{ require } [\hat{H}_1, \hat{H}_0] \neq 0$$

$\Rightarrow$ $\vec{B}_1(t)$ in x-y plane ("transverse")

To get resonance $\vec{B}_1(t)$ oscillates counterclockwise in x-y plane (follows Larmor precession of $\vec{B}_0$)



$$\vec{B}_1 = B_{\perp}\left(\cos(\phi-\omega t)\,\vec{e}_x + \sin(\phi-\omega t)\,\vec{e}_y\right)$$

$$\Rightarrow \hat{H}_1(t) = -\frac{\gamma\hbar B_{\perp}}{2}\left(\cos(\phi-\omega t)\,\hat{\sigma}_x + \sin(\phi-\omega t)\,\hat{\sigma}_y\right)$$

$$= -\frac{\hbar\Omega}{2}\left(\hat{\sigma}_+ e^{i\phi} e^{-i\omega t} + \hat{\sigma}_- e^{-i\phi} e^{+i\omega t}\right)$$

where $\Omega \equiv \gamma B_{\perp} \equiv$ Rabi frequency

Note $\hat{\sigma}_+ = |+_z\rangle\langle -_z|$ "absorption" $e^{-i\omega t}$

$\hat{\sigma}_- = |-_z\rangle\langle +_z|$ "emission" $e^{+i\omega t}$

$$\Rightarrow i\hbar \frac{\partial}{\partial t}|\tilde\Psi(t)\rangle = \underbrace{\left(\hat D \hat H \hat D^\dagger + i\frac{\partial \hat D}{\partial t}\hat D^\dagger\right)}_{\text{New Hamiltonian } \tilde H} |\tilde\Psi(t)\rangle$$

Aside $\hat D \hat\sigma_\pm \hat D^\dagger = e^{\pm i\omega t}\hat\sigma_\pm$

$$\tilde H = \tilde H_0 + \tilde H_1 - \frac{\hbar\omega}{2}\hat\sigma_z$$ (Going to rotating frame like adding $\vec B$ field • Larmor's theorem)

$$\tilde H_0 = H_0 = \frac{\hbar\omega_0}{2}\sigma_z$$

$$\tilde H_1 = -\frac{\hbar\Omega}{2}\left(\hat\sigma_+ e^{i\phi} + \hat\sigma_- e^{-i\phi}\right)$$

$$= -\frac{\hbar\Omega}{2}\left(\hat\sigma_x\cos\phi + \hat\sigma_y\sin\phi\right)$$ Static

$$\Rightarrow \boxed{\tilde H = -\frac{\hbar}{2}\left(\Delta\hat\sigma_z + \Omega(\cos\phi\,\hat\sigma_x + \sin\phi\,\hat\sigma_y)\right) \quad \text{where } \Delta \equiv \omega - \omega_0 \equiv \text{"Detuning"}}$$

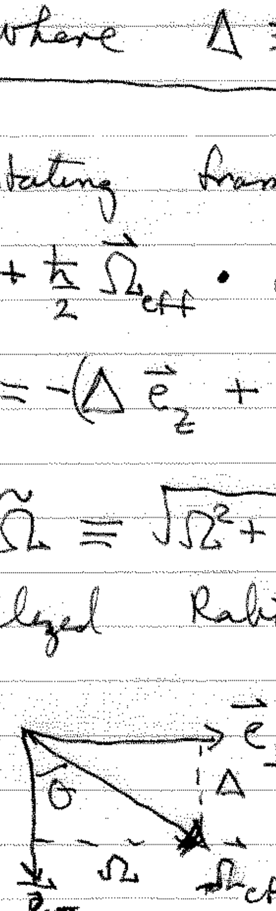
⇒ In rotating frame, new static field interaction

$$\tilde H = +\frac{\hbar}{2}\vec\Omega_{eff}\cdot\vec\sigma$$

$$\vec\Omega_{eff} = -(\Delta\vec e_z + \Omega\vec e_\perp) \equiv \tilde\Omega\,\vec e_n$$

$$\tilde\Omega \equiv \sqrt{\Omega^2+\Delta^2} \qquad \vec e_n = -\left(\frac{\Delta}{\tilde\Omega}\vec e_z + \frac{\Omega}{\tilde\Omega}\vec e_\perp\right)$$

"Generalized Rabi freq."



$$\tan\theta = \frac{\Omega}{\Delta}$$

Solve for evolution

Given $|\Psi(0)\rangle = |-\rangle_z$ $\qquad \hat{H}(t) = \hat{H}_0 + \hat{H}_1(t)$

Solution # 1: Time dependent perturbation theory

Given $\hat{H}_1(t) = \hat{\mathcal{H}}_{abs} e^{-i\omega t} + \hat{\mathcal{H}}_{emis} e^{i\omega t}$

Fermi's Golden Rule

Transition rate: $R_{abs} = \frac{2\pi}{\hbar} |\langle +_z | \hat{\mathcal{H}}_{abs} | -_z \rangle|^2 \mathcal{D}(\omega)$

Density of states

$$R_{abs} = \frac{2\pi}{\hbar}\left(\frac{\hbar^2 \Omega^2}{4}\right)\mathcal{D}(\omega) = \frac{\pi \hbar \Omega^2}{2}\mathcal{D}(\omega)$$

"Incoherent" jump from $|-_z\rangle \Rightarrow |+\rangle_z$

This is not the full story. Applicable for
- Incoherent, broad-band source

and/or
- Final state is "broad"

Isolated two-level system can be solve exactly (not in pert

Solution # 2 Rabi flopping

Step # 1 - Make unitary transformation to remove the time dependence

Go to frame rotating at ω with $\vec{B}_1$

$$\hat{D}(t) = e^{\frac{i\omega t}{2}\hat{\sigma}_z}$$ (rotating counter-clockwise around z-axis)

In new frame: (like interaction picture)

$$|\tilde{\Psi}\rangle = \hat{D}(t)|\Psi_s(t)\rangle \qquad \tilde{A}(t) = \hat{D}^\dagger \hat{A}_s \hat{D}$$

Solution (in rotating frame): Larmor precession about $\vec{e}_n$ by freq $\tilde{\Omega}$ = Rabi flopping

$$\tilde{H} = \frac{\hbar\tilde{\Omega}}{2}\hat{\sigma}_n \qquad \hat{\sigma}_n = \vec{e}_n\cdot\vec{\sigma}$$

$$= -\frac{\Delta}{\tilde{\Omega}}\hat{\sigma}_z - \frac{\Omega}{\tilde{\Omega}}\left(e^{i\phi}\hat{\sigma}_+ + e^{-i\phi}\hat{\sigma}_-\right.$$

$$|\tilde{\Psi}(t)\rangle = e^{-i\frac{\tilde{H}}{\hbar}t}\,|\tilde{\Psi}(0)\rangle$$

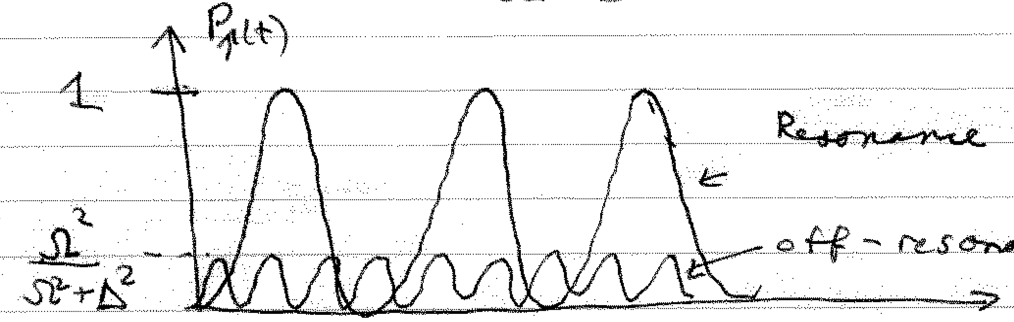
$$= e^{-i\frac{\tilde{\Omega}t}{2}}\,|\downarrow\rangle \qquad \left(|\downarrow\rangle = |-_z\rangle \quad |\uparrow\rangle = |+_z\rangle\right)$$

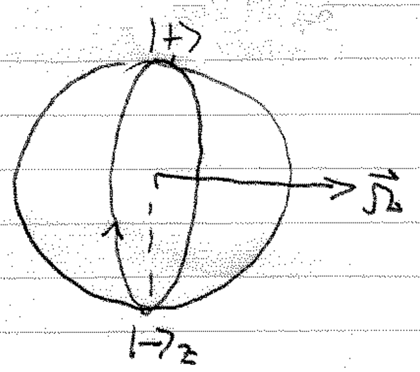
$$= \left(\cos\left(\tfrac{\tilde{\Omega}t}{2}\right)\hat{\mathbb{1}} - i\sin\left(\tfrac{\tilde{\Omega}t}{2}\right)\hat{\sigma}_n\right)|\downarrow\rangle$$

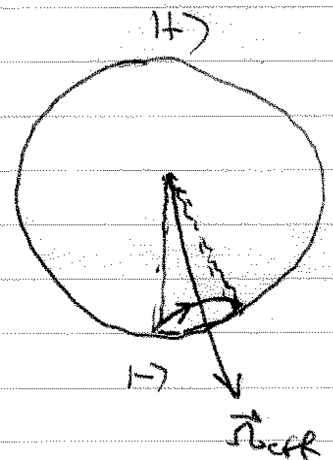
$$\boxed{|\tilde{\Psi}(t)\rangle = \left[\cos\left(\tfrac{\tilde{\Omega}t}{2}\right) - i\frac{\Delta}{\tilde{\Omega}}\sin\left(\tfrac{\tilde{\Omega}t}{2}\right)\right]|\downarrow\rangle + \left[i e^{i\phi}\frac{\Omega}{\tilde{\Omega}}\sin\left(\tfrac{\tilde{\Omega}t}{2}\right)\right]|\uparrow\rangle}$$

Rabi flopping

$$P_\uparrow(t) = |c_\uparrow(t)|^2 = \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left(\frac{\tilde{\Omega}t}{2}\right) = \frac{\Omega^2}{\Omega^2+\Delta^2}\left(\frac{1-\cos(\tilde{\Omega}t)}{2}\right)$$







off resonance

On resonance

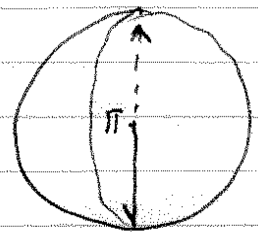
Resonance: $\Delta = 0$

$$|\tilde{\Psi}(t)\rangle = \cos\frac{\Omega t}{2}\,|\downarrow\rangle + i e^{i\phi}\sin\frac{\Omega t}{2}\,|\ \rangle$$

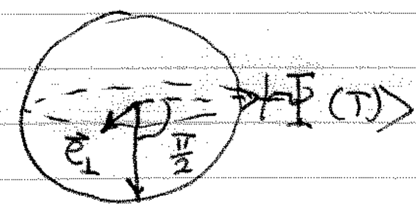
Rotation about $\vec{e}_\perp$ by $\Theta = \Omega t$

$$T = \frac{\Theta}{\Omega} \Rightarrow \text{"}\Theta\text{" pulse}$$

π-pulse: $|\tilde{\Psi}(T)\rangle = i e^{i\phi}\,|\uparrow\rangle = |\uparrow\rangle$



$\pi/2$-pulse $|\tilde{\Psi}(T)\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\rangle + i e^{i\phi}\,|\uparrow\rangle\right)$



equal superposition of $|\downarrow\rangle$ and $|\uparrow\rangle$

Phase set by phase of oscillating $\vec{B}_\perp$

2π-pulse $|\Psi(t)\rangle = -1\,|\downarrow\rangle = -1\,|\Psi(0)\rangle$

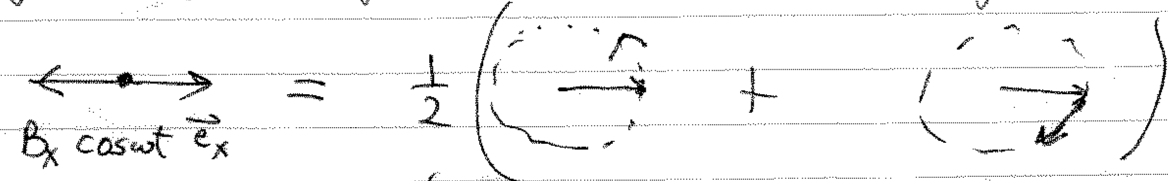
$$\simeq |\downarrow\rangle$$

The overall -1 is the famous phase factor that distinguishes $SU(2)$ from $SO(3)$

No physical effect for ~~a~~ one spin, but important for more than one.

Rotating wave approximation

Typically apply oscillating field along 1D (say x)



$B_x \cos\omega t\, \vec{e}_x$

$$= \frac{B_x}{2}\underbrace{\left(\cos\omega t\, \vec{e}_x + \sin\omega t\, \vec{e}_y\right)}_{\text{co-rotating (resonant)}} + \frac{B_x}{2}\left(\cos\omega t\, \vec{e}_x - \sin\omega t\, \vec{e}_y\right)$$

counter-rotating (anti-resonant)

Rotating wave approximation
$\Rightarrow$ Neglect counter rotating terms

$$\hat{H}_{eff} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z - \frac{\hbar}{2}(\gamma B_x)(\hat{\sigma}_+ + \hat{\sigma}_-)\cos\omega t$$

RWA: $$\hat{H}_{eff} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z - \frac{\hbar}{2}\left(\frac{\gamma B_x}{2}\right)\left(\hat{\sigma}_+ e^{-i\omega t} + \hat{\sigma}_- e^{+i\omega t}\right)$$

Rabi freq $\Omega = \gamma B_1 = \frac{\gamma B_x}{2}$ (only half amplitude contributes)

Note: The RWA is equivalent to time averaging over oscillation ω

Valid to neglect counter-rotating terms only when
$$|\Delta| \ll \omega \quad \text{(Near resonance)}$$

More exact: Floquet Solutions