

Lecture 8: Spontaneous emission: Wigner - Weiskopf

An atom in an excited state is not in a stationary state. We know the atom will eventually decay to the ground state, spontaneously emitting a photon

$$|e\rangle \xrightarrow{\quad} \underbrace{|g\rangle}_{\quad} \rightsquigarrow h\nu$$

The nature of this evolution is due to the coupling of the atom to the quantum electromagnetic field. The idea of spontaneous emission goes back to Einstein and his derivation of the Planck Spectrum from a principle of detailed balance. The rate of spontaneous emission is still known as the "Einstein A-coefficient". Many of the properties can be understood through classical radiative theory - we will see this in homework. There are many subtleties which we will explore throughout the class. One possible particular feature is that the evolution is irreversible, a problem we will study in more general contexts later on.

Formulation:

$$\text{Hamiltonian: } \hat{H} = \hat{H}_A + \hat{H}_V + \hat{H}_{AV}$$

- 2-level atom: $\hat{H}_A = \frac{\hbar\omega_0}{2} (|e\rangle\langle e| + |g\rangle\langle g|) = \frac{\hbar\omega_0}{2} \hat{\sigma}_z^2$

- Vacuum: $\hat{H}_V = \sum_{k,\lambda} \hbar\omega_{k,\lambda} (\hat{a}_{k,\lambda}^\dagger \hat{a}_{k,\lambda} + \text{c.c.})$ neglect

- Interaction: $\hat{H}_{AV} = -\vec{d} \cdot \vec{E}(\vec{R})$

$$\vec{d} = \deg(\hat{\sigma}_+ + \hat{\sigma}_-)$$

$$\hat{\vec{E}}(\vec{R}) = \vec{E}^{(+)}(\vec{R}) + \vec{E}^{(-)}(\vec{R})$$

$$\vec{E}^{(+)}(\vec{R}) = \sum_{\vec{k}, \lambda} i \sqrt{\frac{2\pi k \omega_k}{V}} e^{i \vec{k} \cdot \vec{R}} \hat{a}_{\vec{k}, \lambda} \vec{e}_{\vec{k}, \lambda}$$

$$\hat{J} \cdot \hat{\vec{E}} = (\vec{J}_{\text{cg}} (\hat{\sigma}_+ + \hat{\sigma}_-)) \cdot (\vec{E}^{(+)} + \vec{E}^{(-)})$$

$$= \sum_{\vec{k}, \lambda} \pm (g_{k, \lambda} \hat{\sigma}_+ \hat{a}_{\vec{k}, \lambda} + g_{k, \lambda}^* \hat{\sigma}_- \hat{a}_{\vec{k}, \lambda}^\dagger)$$

$$+ \sum_{\vec{k}, \lambda} \pm (g_{k, \lambda} \hat{\sigma}_- \hat{a}_{\vec{k}, \lambda} + g_{k, \lambda}^* \hat{\sigma}_+ \hat{a}_{\vec{k}, \lambda}^\dagger)$$

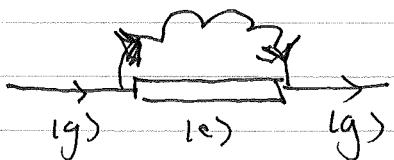
where $\hbar g_{k, \lambda} = -(\vec{J}_{\text{cg}} \cdot \vec{E}_{\vec{k}, \lambda} \sqrt{\frac{2\pi k \omega_k}{V}}) e^{i \vec{k} \cdot \vec{R}}$ (coupling constant)

Feynman picture: Elementary processes

Absorption: $\hat{\sigma}_+ \hat{a}_{\vec{k}, \lambda} \xrightarrow[\text{(g) } 1e]{\gamma} \text{"Co-rotating term in R.W.A"}$

Emission: $\hat{\sigma}_- \hat{a}_{\vec{k}, \lambda}^\dagger \xrightarrow[1e \downarrow \text{(g)}]{} \text{"Co-rotating term in R.W.A"}$

"Counter-rotating terms" important for nonlinear processes: "Virtual states" (off the "mass-shell")



"emission and absorption off a virtual photon"

Contributes to Lamb shift

Spontaneous emission is a resonant phenomenon
 \rightarrow Make RWA

$$\hat{H} = \frac{\hbar\omega_0}{2} \hat{O}_z + \sum_{k,\lambda} \hbar\omega_{k,\lambda} \hat{a}_{k,\lambda}^\dagger \hat{a}_{k,\lambda} + \sum_{k,\lambda} \hbar(g_{k\lambda} \hat{O}_+ \hat{a}_{k\lambda} + g_{k\lambda}^* \hat{a}_{k\lambda}^\dagger \hat{O}_-)$$

Note: With a quantized E_0 field, \hat{H} is not explicitly time dependent. Transitions occur because initial system is not in a stationary state of total H .

Go to the interaction picture:

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}$$

$$\hat{H}_0 = \hat{H}_A + \hat{H}_V, \quad \hat{H}_{int} = \hat{H}_{AV}$$

$$|\Psi^{(I)}_{(+)}\rangle = \hat{U}_0^\dagger_{(+)} |\Psi^{(S)}_{(+)}\rangle \quad \hat{A}^{(I)}_{(+)} = \hat{U}_0^\dagger_{(+)} \hat{A}^{(S)} \hat{U}_0_{(+)}$$

$$\hat{U}_0(t) = e^{-i\hat{H}_0 t/\hbar}$$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} |\Psi^{(I)}_{(+)}\rangle = \hat{H}_{int}^{(I)}_{(+)} |\Psi^{(I)}_{(+)}\rangle$$

$$\hat{H}_{int}^{(I)}(t) = + \sum_{k,\lambda} \hbar(g_{k\lambda} \hat{O}_+ \hat{a}_{k\lambda} e^{i(\omega_{k\lambda} - \omega_q)t} + h.c.)$$

$$|\Psi^{(I)}(t)\rangle = T \underbrace{\left\{ e^{-i \int_0^t \hat{H}_{int}^{(I)}(t') dt'} \right\}}_{\text{time ordered product}} |\Psi^{(I)}(0)\rangle$$

Solve using a decomposition in the basis

$$\left\{ |g, \{n_{k,\lambda}\}\rangle, |e, \{n_{k,\lambda}\}\rangle \right\}$$

↑ ↑
ground state Fock state excited state

At $t=0$ $|\Psi(0)\rangle = |e, 0\rangle$, vacuum

Note: Since the initial state has only one excitation in the RWA the number of excitations is conserved, so we have only possible basis states $|e, 0\rangle, |g, 1_{k,\lambda}\rangle$ one photon in mode

Thus in the interaction picture

$$|\Psi^{(I)}(t)\rangle = c_{e,0}(t) |e, 0\rangle + \sum_{k,\lambda} c_{g,1_{k,\lambda}} |g, 1_{k,\lambda}\rangle$$

Equations of motion for probability amplitudes:

Infinite collection of coupled equations

$$\begin{cases} \dot{c}_{e,0} = -i \frac{\langle e, 0 | \hat{H}_{int}^{(I)}(t) | \Psi^{(I)}(t) \rangle}{\hbar} = -i \sum_{k,\lambda} g_{k,\lambda} e^{-i(\omega_k - \omega_g)t} c_{g,1_{k,\lambda}} \\ \dot{c}_{g,1_{k,\lambda}} = -i \frac{\langle g, 1_{k,\lambda} | \hat{H}_{int}^{(I)}(t) | \Psi^{(I)}(t) \rangle}{\hbar} = -i g_{k,\lambda}^* e^{i(\omega_k - \omega_g)t} c_{e,0} \end{cases}$$

Formal Solution: $c_{g,1_{k,\lambda}}(t) = -i g_{k,\lambda}^* \int_0^t e^{i(\omega_k - \omega_g)t'} c_{e,0}(t') dt'$

$$\Rightarrow \boxed{\dot{c}_{e,0} = - \sum_{k,\lambda} |g_{k,\lambda}|^2 \int_0^t e^{-i(\omega_k - \omega_g)(t-t')} c_{e,0}(t') dt'}$$

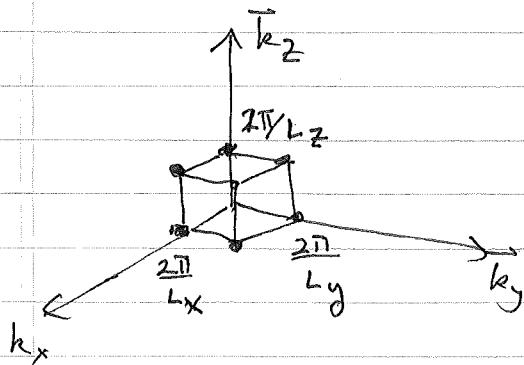
Perturbation theory: $c_{e,0}(0) \approx 1 \Rightarrow$ Fermi's Golden Rule

Wigner-Weisskopf approx: $c_{e,0}(t)$ Slowly varying

\Rightarrow Markov approx

Going to the continuum \Rightarrow Density of states

k -space: Mode quantization in "box" with periodic b.c.'s. $k_i = n_i \frac{2\pi}{L_i}$



$$\frac{1 \text{ mode}}{(\frac{2\pi}{L_x})(\frac{2\pi}{L_y})(\frac{2\pi}{L_z})} = \frac{V}{(2\pi)^3}$$

$= D(\vec{k}) = \text{density of modes in } \vec{k}-\text{space}$

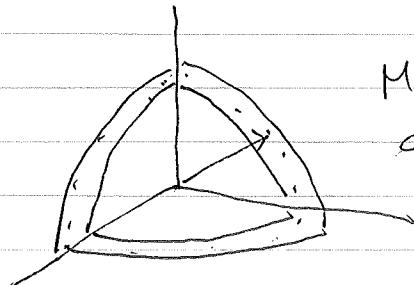
$$\sum_{\vec{k}} \Rightarrow \int d^3k D(\vec{k})$$

$$= \int k^2 d(\vec{k}) dk D(\vec{k})$$

$$= \int d(\vec{k}) d\omega_k \frac{\omega_k^2}{c^3} D(\vec{k})$$

$$= \int d(\vec{k}) d\omega_k D(\omega_k)$$

$$D(\omega_k) = \frac{V}{(2\pi)^3} \frac{\omega_k^2}{c^3} = \text{Density of modes in Frequency}$$



Mode density in a shell in k -space

ω^2 dependence from 3D Space

In continuum limit

$$\dot{C}_{e,0} = - \int_0^\infty d\omega_k \overline{g^2(\omega_k)} \partial(\omega_{k0}) \int_0^t dt' e^{-i(\omega_k - \omega_{eg})(t-t')} c_{e0}(t')$$

where $\overline{g^2(\omega_k)} = \sum_\lambda \int d(\vec{k}) |g_{\vec{k},\lambda}|^2$

Averaged over
directions of
emission and polarization

$$= \frac{1}{\hbar^2} \left(\frac{2\pi\hbar\omega_k}{V} \right) \int d(\vec{k}) \sum_\lambda |\vec{d}_{eg} \cdot \vec{E}_{\vec{k},\lambda}|^2$$

Aside:

$$\int d(\vec{k}) \Rightarrow \sum_\lambda |\vec{d}_{eg} \cdot \vec{E}_{\vec{k},\lambda}|^2 = \sin^2\theta |\vec{d}_{eg}|^2$$

dipole emission pattern

$$\Rightarrow \int d(\vec{k}) \sum_\lambda |\vec{d}_{eg} \cdot \vec{E}_{\vec{k},\lambda}|^2 = \underbrace{\int 2\pi d(\cos\theta)}_{\text{solid angle } d(\vec{k})} \sin^2\theta |\vec{d}_{eg}|^2$$

$$\int d(\vec{k}) \sin^2\theta = 2\pi \int_1^1 d\mu (1-\mu^2) = \frac{8\pi}{3}$$

$$\Rightarrow \boxed{\overline{g^2(\omega_k)} = \frac{16\pi^2}{3\hbar V} \omega_k^2 |\vec{d}_{eg}|^2}$$

(Next Page)

Wigner - Weisskopf approximation

Weak coupling of atom to vacuum ($\alpha = \frac{e^2}{4\pi c} = \frac{1}{157} \ll 1$)

$\Rightarrow C_{eo}(t)$ varies slowly on the time scale $\frac{1}{\omega_{eg}}$

Rate of variation of C_{eo} is Γ to be determined

\Rightarrow Can take $C_{eo}(t')$ to be approximately constant over time $t > \omega_{eg}$ but $t \ll \Gamma^{-1}$

"Coarse graining"

$$\dot{C}_{eo} \approx - \left[\int_0^\infty d\omega_k \bar{g}(\omega_k) D(\omega_k) \int_0^t dt' e^{-i(\omega_k - \omega_{eg})(t-t')} \right] C_{eo}(t)$$

This is a Markov-approximation. Dynamics of C_{eo} depends only on time t and not history for $t' < t$; "no memory of past"

Now $t \rightarrow \infty$ compared to $\frac{1}{\omega_{eg}}$ (still small compared to Γ)

$$\dot{C}_{eo} = - \left[\int_0^\infty d\omega_k \bar{g}(\omega_k) D(\omega_k) (-i \zeta(\omega_k - \omega_{eg})) \right] C_{eo}(t)$$

where $\zeta(\omega) = +i \int_0^\infty dt e^{-i\omega t}$

Fourier transform of
a step function

(8.8)

$\delta(\omega)$ is a distribution. First regularize:

$$\begin{aligned}\delta(\omega) &= \lim_{\epsilon \rightarrow 0^+} +i \int_0^\infty dt e^{-i\omega t - \epsilon t} \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{+i}{i\omega + \epsilon} = \lim \left\{ \frac{-\omega}{\omega^2 + \epsilon^2} + i \frac{\epsilon}{\omega^2 + \epsilon^2} \right\}\end{aligned}$$

"Complex Lorentzian"



$$\lim_{\epsilon \rightarrow 0^+} (\text{Cauchy Principle part}) = \underbrace{P\left(\frac{1}{\omega}\right)}_{\text{Cauchy Principle part}} + i\pi \underbrace{\delta(\omega)}_{\text{Dirac delta}}$$

Cauchy Principle part Dirac delta

$$\Rightarrow P \int_{-a}^b \frac{f(x)}{x} dx = \lim_{\epsilon \rightarrow 0^+} \left[\int_{-a}^{\epsilon} \frac{f(x)}{x} + \int_{-\epsilon}^b \frac{f(x)}{x} \right] dx$$

\Rightarrow In Markov approximation

$$\dot{c}_{e0} = - \left[\int d\omega_k \overline{g(\omega_k)} D(\omega_k) \left(\pi \delta(\omega_k - \omega_{eg}) - i P\left(\frac{1}{\omega_k - \omega_{eg}}\right) \right) \right] c_{e0}^{(t)}$$

$$= - \left(\frac{\Gamma}{2} - iS \right) c_{e0} \Rightarrow \boxed{c_{e0}(t) = e^{-\frac{\Gamma}{2}t} e^{iSt} c_{e0}^{(0)}}$$

Probability decay rate

$$\Gamma = 2\pi \overline{g(\omega_k)} D(\omega_k) = \frac{4}{3} |\vec{d}_{eg}|^2 \frac{\omega^3}{\hbar C^3} \quad \begin{array}{l} \text{(Fermi's} \\ \text{Golden rule)} \end{array}$$

$$\boxed{\Gamma = \frac{4}{3} \alpha |e| \vec{x} |g|^2 \frac{\omega^3}{C^2} \quad \alpha = \frac{e^2}{\hbar C}}$$

Einstein A coeff

$$S = -P \int_0^\infty \frac{\overline{g(\omega_k)} \partial D(\omega_k)}{\omega_k - \omega_{\text{eg}}} d\omega_k$$

(logarithmic divergence)

One contribution to the Lamb Shift

- * RWA
- * Dipole approximation
- * Multilevel atom
- * Relativity (finite cut-off at mc^2/\hbar)
- * Self-energy effects

However: Emission is a resonant effect

⇒ RWA and dipole approx are excellent

Note: $\dot{c}_{e,0} = -\left(\frac{\Gamma}{2} - i\delta\right)c_{e,0}$ is irreversible

How did we start with the unitary, time reversible Schrödinger equation and end with ~~the~~ irreversible behavior?

The "slide of hand" occurred at the Markov approximation. By throwing away memory effects, we throw away reversible behavior. We will return for a much closer look at this problem later in the course.

(8.10)

Given the solution $C_{e,0}(t) = C_{e,0}(0) e^{-\frac{\Gamma}{2}t - i\delta t}$
 we can find $C_g(t)$.

- Firstly, we ~~absorb~~ absorb δ into ω_{eg} to get a renormalized energy $\omega_{eg} \Rightarrow \omega_{eg} - \delta$. ~~which~~
- Next we look at the solution with the atom initially in the excited state in the vacuum

$$C_{e,0}(0) = 1$$

$$\Rightarrow \dot{C}_{g,1_{k\lambda}} = -ig_{k,\lambda}^* \int_0^t e^{i(\omega_k - \omega_{eg})t'} C_{e,0}(t') dt'$$

$$\dot{C}_{g,1_{k\lambda}} \approx -ig_{k,\lambda}^* \int_0^t e^{(i(\omega_k - \omega_{eg}) - \frac{\Gamma}{2})t'} dt'$$

$$\Rightarrow C_{g,1_{k\lambda}}(t) \stackrel{\approx}{=} -ig_{k,\lambda}^* \frac{e^{i(\omega_k - \omega_{eg})t} e^{-\frac{\Gamma}{2}t} - 1}{i(\omega_k - \omega_{eg}) - \frac{\Gamma}{2}}$$

Thus we have the (irreversible) evolution of the pure state of the system atom+field

$$|\Psi\rangle = e^{-\frac{\Gamma}{2}t} |e, 0\rangle$$

$$+ \sum_{k,\lambda} g_{k,\lambda}^* \frac{e^{i(\omega_k - \omega_{eg})t} e^{-\frac{\Gamma}{2}t} - 1}{-(\omega_k - \omega_{eg}) - i\frac{\Gamma}{2}} |g, 1_{k\lambda}\rangle$$

Note: For long times, $t \gg \frac{1}{\Gamma}$, the atom is in the ground state, and the field has one photon in the state

$$|\psi_1\rangle = \sum_{\vec{k}, \lambda} g_{\vec{k}, \lambda}^* \frac{1}{(c\vec{k} - \omega_{eg}) + i\frac{\Gamma}{2}} |1_{\vec{k}, \lambda}\rangle$$

where $g_{\vec{k}, \lambda}^*(\vec{R}) = g_{\vec{k}, \lambda}(0) e^{-i\vec{k} \cdot \vec{R}}$ position of atom

This is a single photon wave packet with momentum distribution.

$$g_{\vec{k}, \lambda} = g_{\vec{k}, 0} \frac{e^{-i\vec{k} \cdot \vec{R}}}{(c|\vec{k}| - \omega_{eg}) + i\frac{\Gamma}{2}}$$

⇒ Propagating spherical wave in space with wave function

$$E(x) = \frac{(ic\vec{w} - \frac{\Gamma}{2})(t - \frac{|x - \vec{R}|}{c})}{|x - \vec{R}|} \hat{H} \left(t - \frac{|x - \vec{R}|}{c} \right)$$

