Lecture 10: Phase-Space Representations (I)

We saw in Lecture 9 the displacement operators

$$\hat{D}(\alpha = Q + iP) = e^{i\hat{P}Q - i\hat{Q}P} = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$$

These operators form a group of translations in phase-space:

$$\hat{D}(\alpha_1)\,\hat{D}(\alpha_2) = \hat{D}(\alpha_1 + \alpha_2)\, e^{i\Theta(\alpha_1,\alpha_2)}$$

$$\Theta(\alpha,\beta) = \mathrm{Im}(\alpha_1\alpha_2^*) = \frac{\alpha_1\alpha_2^* - \alpha_1^*\alpha_2}{2i}$$

This is the "Weyl-Heisenberg" group generated by the canonical algebra

$$[\hat{q},\hat{p}] = i\hbar \qquad [a, a^\dagger] = 1$$

This group is very useful in studying comparisons between quantum and classical dynamics. It is used often in studies of "quantum chaos". For our purposes, $\hat{Q}$ and $\hat{P}$ represent the "quadratures" of the field mode $\hat{a}$. We seek to understand the connections between classical and quantum descriptions for the field.

Fundamental relations for representation

Recall from Lecture, operators for a linear space with inner product

$$(\hat{A} \mid \hat{B}) \equiv \mathrm{Tr}(\hat{A}^\dagger \hat{B})$$

As functions we know that plane waves form a complete orthonormal basis

$$\int \frac{dk}{\sqrt{2\pi}} e^{ik(x-x')} = \delta(x-x')$$

$$f(x) = \int \frac{dk}{\sqrt{2\pi}} \tilde{f}(k) e^{ikx} \quad \text{where} \quad \tilde{f}(k) = \int \frac{dx}{\sqrt{2\pi}} f(x) e^{-ikx}$$

These functional relations can be concisely written in Dirac notation

$$\int dk \; |k\rangle\langle k| = \hat{\mathbb{1}} \qquad \langle x|k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}}$$

$$|f\rangle = \int dk \; |k\rangle\langle k|f\rangle$$

The generalization from plane waves to displacement operators follows analogously. These operators are complete orthonormal

Orthonormality

$$\mathrm{Tr}\left(\hat{D}(\alpha)\hat{D}^\dagger(\beta)\right) = \int d^2\gamma \, \langle\gamma|\hat{D}(\alpha)\hat{D}^\dagger(\beta)|\gamma\rangle$$

$$= \pi\delta^{(2)}(\alpha-\beta) \quad \text{(see Homework)}$$

$$= \pi\delta(Q_\alpha - Q_\beta)\,\delta(P_\alpha - P_\beta)$$

Complete

$$\int \frac{d^2\alpha}{\pi} \operatorname{Tr}\left(\odot\, \hat{D}^\dagger(\alpha)\right) \hat{D}(\alpha) = \hat{\mathbb{1}}$$

Insert any operator here

$$\Rightarrow \boxed{\hat{\mathcal{O}} = \int \frac{d^2\alpha}{\pi} \operatorname{Tr}\left(\hat{\mathcal{O}}\, \hat{D}^\dagger(\alpha)\right) \hat{D}(\alpha)}$$

"Distribution function"

Superoperator notion

"Bra and ket" $|\hat{\mathcal{O}})$ $(\hat{\mathcal{O}}|$

$$\int \frac{d^2\alpha}{\pi} |\hat{D}(\alpha))(\hat{D}(\alpha)| = \hat{\mathbb{1}}$$

$$|\hat{\mathcal{O}}) = \int \frac{d^2\alpha}{\pi} |\hat{D}(\alpha))(\hat{D}(\alpha)|\hat{\mathcal{O}})$$

$$|\hat{\mathcal{O}}) = \int \frac{d^2\alpha}{\pi} |\hat{D}(\alpha)) \operatorname{Tr}(\hat{D}^\dagger(\alpha)\, \hat{\mathcal{O}})$$

This is the Natural generalization of "Fourier's theorem" for operators on an ∞-dim Hilbert space ("Continuous variables")

Question of operator ordering

Consider an operator on phase space $\hat{O}(\hat{a}, \hat{a}^\dagger)$
A natural representation on phase space might by

$$\hat{O}(\hat{a}, \hat{a}^\dagger) \Longrightarrow O(\alpha, \alpha^*)$$

But we must be carefull of operator ordering.
$\hat{a}$ and $\hat{a}^\dagger$ do not commute, whereas α and α^* do.

Formal definition: Power series.

$$\hat{O}(\hat{a}, \hat{a}^\dagger) = \sum_{n,m} C^{(N)}_{n,m} (\hat{a}^\dagger)^n (\hat{a})^m$$

$\Uparrow$ coefficients in "normal order"

$$= \sum_{n,m} C^{(A)}_{n,m} (\hat{a})^n (\hat{a}^\dagger)^m$$

$\Uparrow$ coefficients in "antinormal order"

$$= \sum_{n,m} C^{(S)}_{n,m} \{(\hat{a}^\dagger)^n (\hat{a})^m\}$$

$\Uparrow$ coefficients in "symmetric order"

e.g. $\{\hat{a}^\dagger a\} = \dfrac{\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger}{2}$

$$\{\hat{a}^\dagger \hat{a}^2\} = \frac{1}{3}\left(\hat{a}^\dagger \hat{a}^2 + \hat{a}\hat{a}^\dagger \hat{a} + \hat{a}^2 \hat{a}^\dagger\right)$$

Average over all possible orders

We thus have three different phase-space representations

$$\hat{O}^{(N)}(\alpha,\alpha^*) = \sum_{n,m} C^{(N)}_{n,m} (\alpha^*)^n (\alpha)^m \equiv Q_{\hat{O}}(\alpha,\alpha^*)$$

$$\hat{O}^{(A)}(\alpha,\alpha^*) = \sum_{n,m} C^{(A)}_{n,m} (\alpha^*)^n (\alpha)^m \equiv P_{\hat{O}}(\alpha,\alpha^*)$$

$$\hat{O}^{(S)}(\alpha,\alpha^*) = \sum_{n,m} C^{(S)}_{n,m} (\alpha^*)^n (\alpha)^m = W_{\hat{O}}(\alpha,\alpha^*)$$

Questions

(1) Do Q, P, W always exist as well behaved functions (maybe delta function?)

(2) Can we find Q, P, W directly from $\hat{O}$ without the tedious process of finding the power series?

We will pursue the answers to the questions in the next lecture. We will be particularly interested in representations of the density operator. For that case:

- $W_{\hat{\rho}}(\alpha,\alpha^*) \equiv$ "Wigner function"
- $Q_{\hat{\rho}}(\alpha,\alpha^*) =$ "Husimi function"
- $P_{\hat{\rho}}(\alpha,\alpha^*) =$ "Glauber-Sudarshan function" (familiar from Lecture 9)