Lecture 11: Phase-space Representations (II)

In the previous lecture we saw that any operator on an infinite dimensional Hilbert space can be expanded in terms of the phase-space displacement operators, $\hat{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* \hat{a})$

$$\hat{O} = \int \frac{d^2\alpha}{\pi} \, \mathrm{Tr}\left(\hat{O}\,\hat{D}^\dagger(\alpha)\right) \hat{D}(\alpha)$$

$\alpha \to -\alpha$

$$\boxed{\hat{O} = \int \frac{d^2\alpha}{\pi} \, \mathrm{Tr}\left(\hat{O}\,\hat{D}(\alpha)\right) \hat{D}^\dagger(\alpha)}$$

this latter form is the convention

We can use this expression to find the representations of $\hat{O}$ in different operator orderings

First note:

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = \sum_{m=0}^{\infty} \frac{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})^m}{m!}$$

$$= \sum_{n,m=0} \frac{(\alpha)^n (\alpha^*)^m}{n!\, m!} \left\{(\hat{a}^\dagger)^n (\hat{a})^m\right\} \leftarrow \text{symmetric ordering}$$

$$\Rightarrow \hat{D}(\alpha) = \left\{\hat{D}(\alpha)\right\} .$$

The phase-space displacement operator is naturally ~~ordered~~ symmetrically ordered

Now use $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}$ if $\begin{cases} [\hat{A},[\hat{A},\hat{B}]] = 0 \\ [\hat{B},[\hat{A},\hat{B}]] = 0 \end{cases}$

$$\Rightarrow \hat{D}(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* \hat{a}} = e^{-\frac{|\alpha|^2}{2}} \underbrace{e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}}}$$

Normally ordered $\equiv \hat{D}_{+1}(\alpha)$

$$= e^{\frac{|\alpha|^2}{2}} \underbrace{e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger}}$$

Antinormally ordered $= \hat{D}_{-1}(\alpha)$

$$\Rightarrow \hat{D}_{+1}(\alpha) = e^{+\frac{|\alpha|^2}{2}} \hat{D}(\alpha) \quad (\text{usually written } :\hat{D}(\alpha):)$$

$$\hat{D}_{-1}(\alpha) = e^{-\frac{|\alpha|^2}{2}} \hat{D}(\alpha) \quad \text{anti-normal}$$

More generally: $\hat{D}_\sigma(\alpha) \equiv e^{\sigma \frac{|\alpha|^2}{2}} \hat{D}(\alpha) \quad -1 \le \sigma \le +1$

See Cahill + Glauber Phy Rev 117 1857 (1969)

$\Rightarrow$ We have a natural way of writing $\hat{O}(\hat{a}, \hat{a}^\dagger)$ as an ordered product

$$\hat{O}_\sigma(\hat{a}, \hat{a}^\dagger) = \int \frac{d^2\beta}{\pi} \mathrm{Tr}\left(\hat{O} \hat{D}(\beta) e^{-\sigma \frac{|\beta|^2}{2}}\right) \hat{D}_\sigma^\dagger(\beta)$$

$$= \int \frac{d^2\beta}{\pi} \mathrm{Tr}\left(\hat{O} \hat{D}_{-\sigma}(\beta)\right) \hat{D}_\sigma^\dagger(\beta)$$

This expansion will give a continuum of representations parameterized by σ

Given these integral expressions, we have a way of writing the ordered power series:

e.g. $\hat{\mathcal{O}}^{(N)}(\hat{a},\hat{a}^\dagger) = \sum_{n,m}^{'} c^{(N)}_{n,m} (\hat{a}^\dagger)^n (\hat{a})^m = \int \frac{d^2\beta}{\pi} \mathrm{Tr}\left(\hat{\mathcal{O}} \hat{D}_{-1}(\beta)\right) \hat{D}^\dagger_{+1}(\beta)$

$$= \int \frac{d^2\beta}{\pi} \mathrm{Tr}\left(\hat{\mathcal{O}} e^{-\beta^* \hat{a}} e^{\beta \hat{a}^\dagger}\right) e^{-\beta \hat{a}^\dagger} e^{\beta^* \hat{a}}$$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{O}^{(N)}(\alpha,\alpha^*) &= \mathcal{O}^{(N)}(\hat{a}=\alpha,\ \hat{a}^\dagger=\alpha^*) \\ &= \int \frac{d^2\beta}{\pi} \tilde{\mathcal{O}}^{(N)}(\beta,\beta^*) e^{\alpha\beta^* - \beta\alpha^*} \end{aligned}}$$

$$\text{where } \tilde{\mathcal{O}}^{(N)}(\beta,\beta^*) = \mathrm{Tr}\left(\hat{\mathcal{O}} e^{-\beta^* \hat{a}} e^{\beta \hat{a}^\dagger}\right)$$

Note $\frac{\alpha\beta^* - \beta\alpha^*}{2} = i\,\mathrm{Im}(\alpha\beta^*) = -i Q_\alpha P_\beta + i P_\alpha Q_\beta$

$\Rightarrow$ $\mathcal{O}^{(N)}(\alpha,\alpha^*)$ and $\tilde{\mathcal{O}}^{(N)}(\beta,\beta^*)$ are 2-dim Fourier tranform pairs

Generally

$$\mathcal{O}^{(\sigma)}(\alpha,\alpha^*) = \int \frac{d^2\beta}{\pi} \tilde{\mathcal{O}}^{(\sigma)}(\beta,\beta^*) e^{\alpha\beta^* - \beta\alpha^*}$$

$$\tilde{\mathcal{O}}^{(\sigma)}(\beta,\beta^*) = \mathrm{Tr}\left(\hat{\mathcal{O}}(\hat{a},\hat{a}^\dagger) \hat{D}_{-\sigma}(\beta)\right)$$

Application to the density operator: Phase-space distribution functions

For the particular case of the density operator we define:

$$\rho^{(\sigma)}(\alpha,\alpha^*) = \int \frac{d^2\beta}{\pi}\, \chi^{(\sigma)}(\beta,\beta^*)\, e^{\alpha\beta^* - \beta\alpha^*}$$

$$\chi^{(\sigma)}(\beta,\beta^*) = \mathrm{Tr}\left(\hat{\rho}\, \hat{D}_{-\sigma}(\beta)\right)$$

$\equiv$ "Characteristic function"

The characteristic function is very useful as the generating function to calculate correlation functions

e.g. $$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{a}^{\dagger 2}\hat{a}^2)$$

$$= \frac{\partial}{\partial\beta^{*2}}\frac{\partial}{\partial\beta^2}\mathrm{Tr}\left(\hat{\rho}\, e^{\beta\hat{a}^\dagger} e^{-\beta^*\hat{a}}\right)\Big|_{\beta=\beta^*=0}$$

By convention we defined

Wigner distribution: $W(\alpha,\alpha^*) \equiv \frac{1}{\pi}\rho^{(0)}(\alpha,\alpha^*)$

Glauber–Sudarshan: $P(\alpha,\alpha^*) = \frac{1}{\pi}\rho^{(A)}(\alpha,\alpha^*)$

Husimi distribution: $Q(\alpha,\alpha^*) = \frac{1}{\pi}\rho^{(N)}(\alpha,\alpha^*)$

What are the uses of the distribution functions?

In quantum mechanics we are used to the position-space and momentum-space representations. These can be very useful for calculating mean values and interpreting them.

e.g. $\langle \hat{V} \rangle = \int d^3x \; V(x) \, |\psi(x)|^2$

$\langle \hat{T} \rangle = \int dp \; \frac{p^2}{2m} \; |\tilde{\psi}(p)|^2$

These expressions have a natural interpretation as averages of classical functions over a probability distribution. In phase-space things get trickier. We might like to write

$$\langle \hat{O}(\hat{x}, \hat{p}) \rangle = \int dx dp \; O(x,p) \; W(x,p)$$

where $W(x,p)$ is a distribution on phase-space.

The problem, of course, is that $\hat{x}$ and $\hat{p}$ don't commute, so there are issues of operator ordering. This classical picture is too naive. Generally $W(x,p)$ may be negative; it is not a probability distribution but is generally refered to as a quasi-probability distribution.

These same considerations carry over to the quantum oscillators representing the modes of the E/M field.

Calculating expectation values using quasi-distributions

One more piece of formalism...

Define Fourier transform of displacement ops:

$$\hat{T}_\sigma(\alpha) \equiv \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha^*\beta} \hat{D}_\sigma(\beta)$$

Also complete, orthonormal

$$\mathrm{Tr}\left(\hat{T}_\sigma(\alpha)\hat{T}^\dagger_{-\sigma}(\beta)\right) = \pi\delta(\alpha-\beta)$$

$$\int \frac{d^2\alpha}{\pi} \mathrm{Tr}\left(\odot \hat{T}_\sigma(\alpha)\right)\hat{T}^\dagger_{-\sigma}(\alpha) = \hat{1}\odot$$

We have order representation:

$$\hat{\rho} = \int \frac{d^2\beta}{\pi} \chi^{(\sigma)}(\beta,\beta^*)\hat{D}^\dagger_\sigma(\beta) = \int \frac{d^2\alpha}{\pi} \rho^{(\sigma)}(\alpha,\alpha^*)\hat{T}_\sigma(\alpha)$$

using fact the $\chi^{(\sigma)} \Leftrightarrow \rho^{(\sigma)}$, $\hat{D} \Leftrightarrow \hat{T}$ } Fourier transform pair

True for any operator $\hat{\mathcal{O}} = \int \frac{d^2\gamma}{\pi} \hat{\mathcal{O}}^{(\sigma)}(\gamma,\gamma')\hat{T}^\dagger_\sigma(\gamma)$

Note $\hat{\mathcal{O}}^{(\sigma)}_{(\alpha,\alpha^*)} = \mathrm{Tr}\left(\hat{\mathcal{O}}\hat{T}_{+\sigma}(\alpha)\right)$

$$\Rightarrow \boxed{\langle\hat{\mathcal{O}}\rangle = \int \frac{d^2\alpha}{\pi} \rho^{(\sigma)}(\alpha,\alpha^*)\hat{\mathcal{O}}^{(-\sigma)}(\alpha,\alpha^*)}$$

Expectation value as average of classical function over quasi probability distribution

Thus:

$$\langle \hat{O} \rangle = \int \frac{d^2\alpha}{\pi} \rho^{(N)}(\alpha,\alpha^*)\, \hat{O}^{(A)}(\alpha,\alpha^*) = \int d^2\alpha\, Q(\alpha)\, \hat{O}^{(A)}_{(\alpha)}$$

$$= \int \frac{d^2\alpha}{\pi} \rho^{(A)}(\alpha,\alpha^*)\, \hat{O}^{(N)}(\alpha,\alpha^*) = \int d^2\alpha\, P(\alpha)\, \hat{O}^{(N)}(\alpha)$$

$$= \int \frac{d^2\alpha}{\pi} \rho^{(S)}(\alpha,\alpha^*)\, \hat{O}^{(S)}(\alpha,\alpha^*) = \int d^2\alpha\, W(\alpha)\, \hat{O}^{(S)}(\alpha)$$

Properties of Q, P, W

- Q-function: Normal order form of $\hat{\rho}$, used to calculate anti-normally ordered expectation values

Note $\rho^{(N)}(\hat{a},\hat{a}^\dagger) = \sum c^{(N)}_{n,m}\, \hat{a}^{\dagger n}\, \hat{a}^m$

$$\Rightarrow \rho^{(N)}(\alpha,\alpha^*) = \sum c^{(N)}_{n,m} \langle\alpha|\hat{a}^{\dagger n}\, \hat{a}^m|\alpha\rangle = \sum c_{n,m} (\alpha^*)^n\, \alpha^m$$

$$\Rightarrow \boxed{Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle}$$ Exists! (Simple way to calculate)

Since $\hat{\rho}$ has positive eigenvalues

$Q(\alpha) \geq 0$: True probability

$$\mathrm{Tr}(\hat{\rho}) = \int d^2\alpha\, \langle\alpha|\hat{\rho}|\alpha\rangle = \pi \int d^2\alpha\, Q(\alpha) = \underline{1}$$

(our normalization convention)

Though Q always exists, we will see that the anti-normal rep of $\hat{O}$ does not so Q is of limited utility in calculating expectations

Aside

$$Q(\alpha,\alpha^*) = \mathrm{Tr}\left(\hat{\rho}\,\frac{|\alpha\rangle\langle\alpha|}{\pi}\right) = \mathrm{Tr}\left(\hat{\rho}\,\hat{T}_1(\alpha)\right)$$

$$\Rightarrow \hat{T}_1(\alpha) = \frac{|\alpha\rangle\langle\alpha|}{\pi}$$

P-function: Antinormal rep of $\hat{\rho}$

$$\hat{\rho} = \int d^2\alpha\, P(\alpha)\,\hat{T}_1(\alpha) = \int \frac{d^2\alpha}{\pi}\, P(\alpha)\,|\alpha\rangle\langle\alpha|$$

Thus, as promised, $P(\alpha)$ is the Glauber-Sudarshan representation, giving a statistical mixture of coherent states

$$\langle\hat{O}\rangle = \mathrm{Tr}\left(\hat{\rho}\,\hat{O}\right) = \int d^2\alpha\, P(\alpha)\,\frac{\langle\alpha|\hat{O}|\alpha\rangle}{\pi}$$

$$= \int d^2\alpha\, P(\alpha)\, O^{(N)}(\alpha)$$

Useful for calculating <u>normally ordered</u> products

Now Characteristic function

$$\chi_\rho(\beta) = e^{|\beta|^2/2}\,\mathrm{Tr}\left(\hat{\rho}\,\hat{D}(\beta)\right) \leq e^{|\beta|^2/2}$$

$\Rightarrow$ χ_ρ can blow up!

$\Rightarrow$ $P(\alpha)$ may not exist even as as "tempered" distribution

Wigner function: Symmetrically ordered form of $\hat{\rho}$

$$\hat{\rho} = \int \frac{d^2\beta}{\pi} \underbrace{\mathrm{Tr}(\hat{\rho}\hat{D}(\beta))}_{\chi_W(\beta)} \hat{D}^\dagger(\beta)$$

$$\mathrm{Tr}(\hat{\rho}^2) = \int \frac{d^2\beta}{\pi} |\chi_W(\beta)|^2 \leq 1$$

$\Rightarrow$ $|\chi_W(\beta)|^2$ is square integrable

$\Rightarrow$ $W(\alpha)$ always exists, but can be negative

The Wigner function is the most useful in a general context since it always exists and one can calculate averages

Relations between P, Q, W

$$W(\alpha) = \frac{2}{\pi} \int P(\beta)\, e^{-2|\beta-\alpha|^2}\, d^2\beta$$

$$Q(\alpha) = \frac{1}{\pi} \int P(\beta)\, e^{-|\alpha-\beta|^2}\, d^2\beta$$

} Gaussian convolutions

$\Rightarrow$ W and Q are "smoothed" versions of P with Q the most coarse grained (least ripples)

Example: Coherent state $|\alpha_0\rangle$ $\quad \rho = |\alpha_0\rangle\langle\alpha_0|$

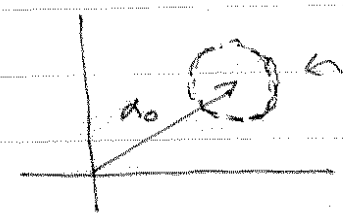
$$\Rightarrow Q(\alpha) = |\langle\alpha|\alpha_0\rangle|^2 = e^{-|\alpha-\alpha_0|^2}$$

$$P(\alpha) = \pi\,\delta^{(2)}(\alpha-\alpha_0)$$

$$W(\alpha) = e^{-2|\alpha-\alpha_0|^2}$$

$$= e^{-2(Q-Q_0)^2}\, e^{-2(P-P_0)^2} \quad \text{Gaussian}$$

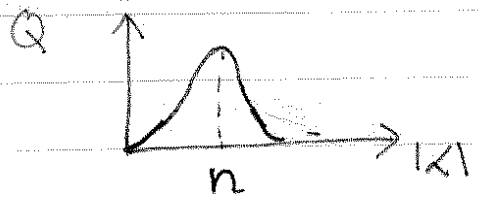
$$\langle Q\rangle = Q_0,\ \langle P\rangle = P_0 \qquad \Delta Q^2 = \Delta P^2 = \frac{1}{4}$$

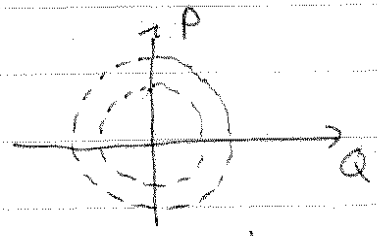


← error contour of Wigner function

Example: Number state $|n\rangle$ $\quad \rho = |n\rangle\langle n|$

- $Q(\alpha) = |\langle\alpha|n\rangle|^2 = e^{-|\alpha|^2}\dfrac{(|\alpha|^2)^n}{n!}$





Contour plot

ring in phase space (no phase, def amplitude)

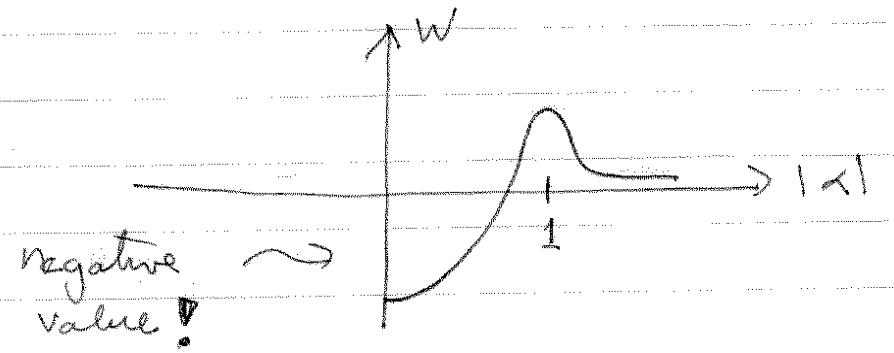
- $W(\alpha) = \int \frac{d^2\beta}{\pi}\, \chi_W(\beta)\, e^{\alpha\beta^* - \beta\alpha^*}$

$$\chi_W(\beta) = \langle n|\hat{D}(\beta)|n\rangle \overset{\text{algebra}}{=} \mathcal{L}_n(|\beta|^2)$$

Laguerre poly

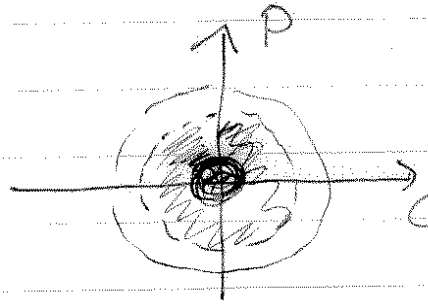
Fourier transform

$$\Rightarrow \quad W_n(\alpha) = 2(-1)^n e^{-2|\alpha|^2} \mathcal{L}_n(4|\alpha|^2)$$



eg $n = 1$

$\Rightarrow$ Fock-state is not quasi-classical



ring peak around $n=1$

with negative dip

P function

$$P(\alpha) = \pi \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial \alpha^n \partial \alpha^{*n}} \delta^{(2)}(\alpha) \qquad \left(\begin{matrix}\text{sal}\\ \text{Scully}\end{matrix}\right)$$

Very singular

Fock state is nothing like a statistical mixture of coherent states