

## Lecture 13: Optical Coherence and Photon Statistics

### Summary of Last Lecture: Classical Statistical Optics

Classically uncertain ("fluctuating field")

$$E(\vec{x}, t) = \sum_k E_0 \underset{\substack{\uparrow \\ \text{Fourier coeff.}}}{\alpha_k} u_k(\vec{x}, t)$$

modes

Fundamental object: Probability distribution

$P(\{\alpha_k\}, t)$  in Fourier modes

• or  $P[E(x)] = \int d^2 \alpha_k \delta(E(x) - \sum_k E_0 \alpha_k u_k(x)) P(\{\alpha_k\})$

Measurements are expectation values

$$\langle f(E) \rangle_t = \int d^2 E P[E, t] f(E)$$

ergodic  $= \frac{1}{T} \int_{-T}^T dt f(E(x, t))$  (time average)

Stationary  $P[E]$ , independent of time

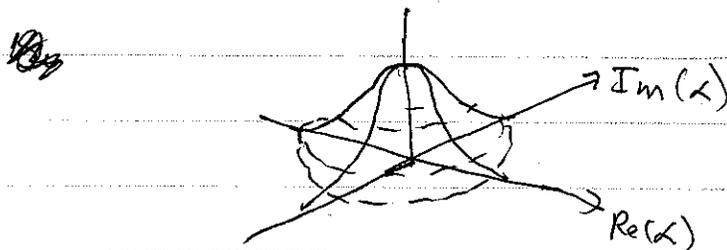
"Natural Light", "Chaotic", "Thermal"

$$P(\{\alpha_k\}) = \prod_k P(\alpha_k) \quad \text{Statistically independent modes}$$

$$P(\alpha_k) = \frac{1}{\pi N} e^{-\frac{|\alpha_k|^2}{N}} \quad \text{Incoherent addition of independently radiating oscillators (collisions)}$$

$$\langle |\alpha_k|^2 \rangle = \int d^2\alpha_k P(\alpha_k) |\alpha_k|^2 = N$$

$$\Rightarrow P(\alpha_k) = \frac{1}{\pi \langle |\alpha_k|^2 \rangle} e^{-\frac{|\alpha_k|^2}{\langle |\alpha_k|^2 \rangle}} \quad \text{Gaussian in phase-space}$$



Since  $P(\alpha_k)$  is only a function of  $|\alpha_k|$  we can ~~and~~ express this as a probability in intensity

$$P(I_k) = \frac{1}{\langle I \rangle} e^{-I_k / \langle I \rangle}$$

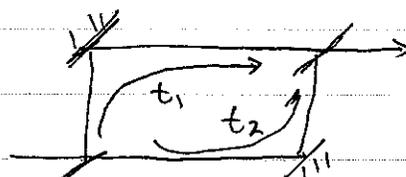
$$\Rightarrow \langle I_k^n \rangle = \int_0^\infty I_k^n e^{-I_k / \langle I \rangle} dI_k = n! \langle I \rangle^n$$

$$\Delta I^2 = \langle I^2 \rangle - \langle I \rangle^2 = 2 \langle I \rangle^2 - \langle I \rangle^2$$

$$\Rightarrow \boxed{\Delta I^2 = \langle I \rangle^2} \quad \text{Fluctuations!}$$

## Correlation, Coherence, and Interference

First order interference: Field-Field correlation



$$I_{\text{out}} \sim G^{(1)}(t_1, t_1) + G^{(1)}(t_2, t_2) + G^{(1)}(t_1, t_2) + G^{(1)}(t_2, t_1)$$

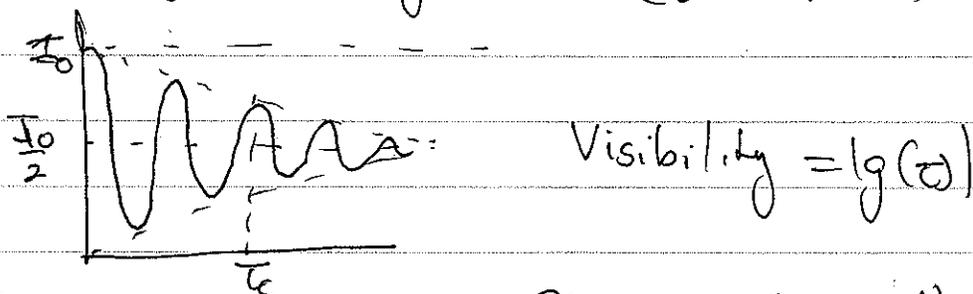
$$G^{(1)}(x_1, x_2) \equiv \langle E^{(-)}(x_1) E^{(-)}(x_2) \rangle$$

Normalized  $g^{(1)}(x_1, x_2) \equiv \frac{G^{(1)}(x_1, x_2)}{\sqrt{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)}}$

Cauchy-Schwarz  $\Rightarrow |g^{(1)}(x_1, x_2)| \leq 1$

Quasi-monochromatic:

$$I_{\text{out}} = \frac{I_0}{2} (1 + |g^{(1)}(\tau)| \cos(\omega_0 \tau + \phi(\tau)))$$

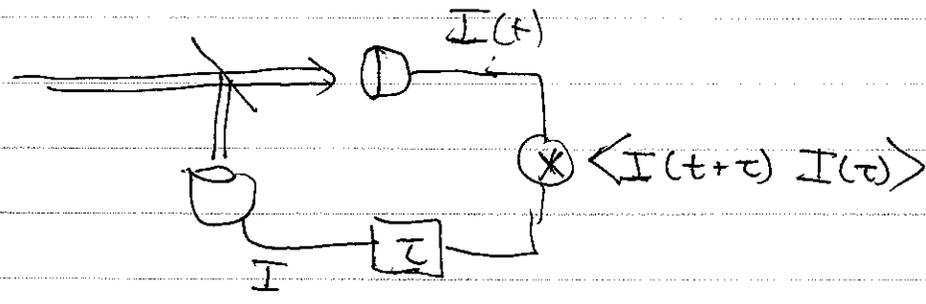


Spectral energy density  $|\tilde{E}(\omega)|^2 = \int \frac{d\tau}{2\pi} G^{(1)}(\tau) e^{i\omega\tau}$

Coherence length  $l_c = c\tau_c \sim \frac{c}{\Delta\omega}$

Over what path difference are the fields "correlated" (have well defined path difference)

Second order interference : Intensity-Intensity correlation (HBT)



$$G^{(2)}(x_1, x_2; x'_1, x'_2) \equiv \langle E^{(-)}(x_1) E^{(-)}(x_2) E^{(+)}(x'_1) E^{(+)}(x'_2) \rangle$$

$$g^{(2)}(x_1, x_2; x'_1, x'_2) \equiv \frac{G^{(2)}(x_1, x_2; x'_1, x'_2)}{\sqrt{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2) G^{(1)}(x'_1, x'_1) G^{(1)}(x'_2, x'_2)}}$$

Cauchy-Schwarz:  $|G^{(2)}(x_1, x_2; x_1, x_2)|^2 \leq G^{(2)}(x_1, x_1; x_1, x_1) G^{(2)}(x_2, x_2; x_2, x_2)$   
 (Classical)

Signal

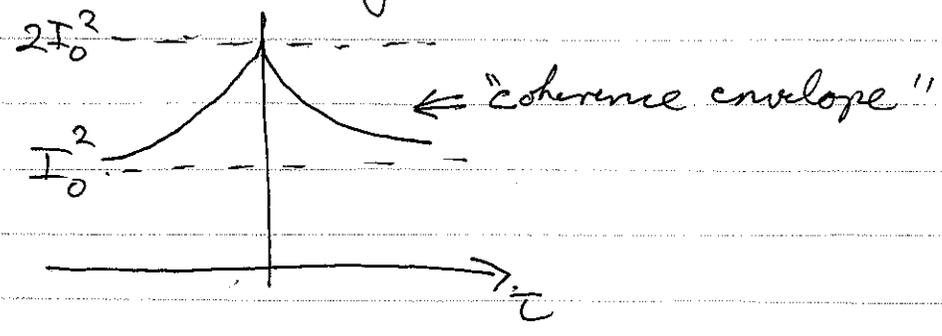
$$\text{HBT } \langle I(t+\tau) I(t) \rangle = \langle E^{(-)}(t+\tau) E^{(-)}(t) E^{(+)}(t+\tau) E^{(+)}(t) \rangle$$

$$= G^{(2)}(t+\tau, t; t+\tau, t)$$

Natural light =  $G^{(1)}(t+\tau, t+\tau) G^{(1)}(t, t) + G^{(1)}(t+\tau, t) G^{(1)}(t, t+\tau)$   
 (Gaussian stats)

$$= I_0^2 + |G^{(1)}(\tau)|^2$$

$$= I_0^2 (1 + |g^{(1)}(\tau)|^2)$$



HBT signal:  $\langle I(t+\tau) I(t) \rangle - \langle I(t+\tau) \rangle \langle I(t) \rangle$   
 $= \langle (I(t+\tau) - \langle I(t+\tau) \rangle) (I(t) - \langle I(t) \rangle) \rangle$

$C(\tau) = \langle \delta I(t+\tau) \delta I(t) \rangle = \langle \delta I(\tau) \delta I(0) \rangle$   
 -  $\uparrow$  Covariance                       $\uparrow$  stationarity

$C(0) = \delta I^2$       Variance : fluctuations

$C(\tau \rightarrow \infty) \rightarrow 0$       No correlation

$\Rightarrow$  Can measure coherence length

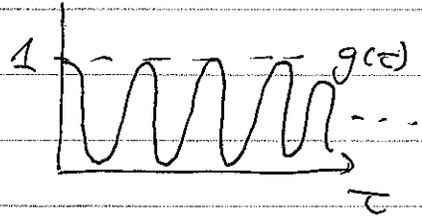
Compare to "coherent source"

e.g.  $P[\epsilon] = \delta(\epsilon - \epsilon_0)$        $\leftarrow$  fixed function

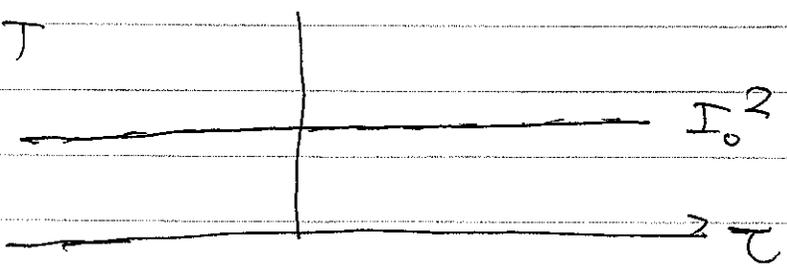
$\Rightarrow G^{(n)}(x_1, x_2 \dots x_n; x'_1, x'_2 \dots x'_n) = E_0^*(x_1) \dots E_0^*(x_n) E_0(x_1) \dots E_0(x_n)$

$\Rightarrow g^{(n)} = 1$       Factorized

Michelson or Mach-Zehnder



HBT



$\langle \delta I(t+\tau) \delta I(t) \rangle = 0 \quad \forall \tau$

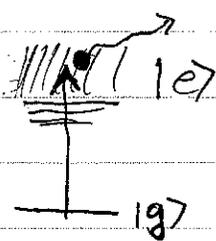
$\Rightarrow$  Statistically independent

Quantum Theory: Glauber 1963

We now turn to study the coherence properties of light described by the quantum theory. Our main goal here is to characterize phenomena which are "nonclassical", i.e. not describable by the classical statistical theory up to this point.

In 1963 R.J. Glauber established the foundation of this subject. When characterizing quantum phenomena a crucial component is measurement. Glauber's major contribution was to consider quantum expressions for the explicitly measured quantities.

Photo detection



Photon counters are initiated by a quantum event: ejection of an electron from some quasi-bound state into the continuum. The electron current is "amplified" to a macroscopic

Modeling the ionization event from a single atom

$$\hat{H}_{int} = -\hat{d} \cdot \hat{E} = -\sum_e d_{ge} (\hat{E}^{(+)} |e\rangle\langle g| + \hat{E}^{(-)} |g\rangle\langle e|)$$

(in RWA and electric dipole approx)

(13.7)

The transition rate to a given final state is determined in lowest order by Fermi's Golden Rule

$$W_{e \leftarrow g} = s |\langle \psi_f | \hat{E}_{(x)}^{(+)} | \psi_i \rangle|^2$$

where  $s$  is the detector "sensitivity" and  $|\psi_i\rangle$ ,  $|\psi_f\rangle$  are the initial/final states of the field

Now since our measurement does not care about the final state of the field

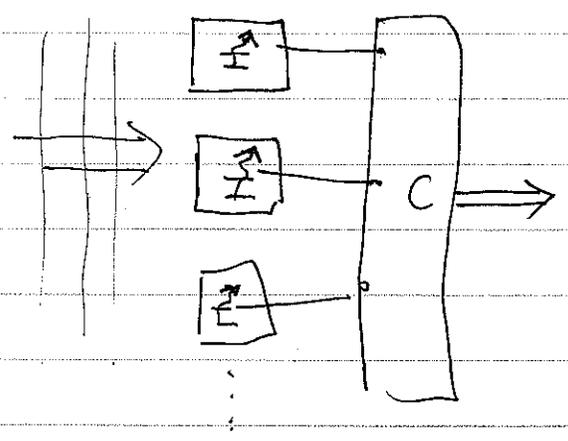
$$\begin{aligned} W &= \sum_f W_{f \leftarrow i} = s \sum_f |\langle \psi_f | \hat{E}_{(x)}^{(+)} | \psi_i \rangle|^2 \\ &= s \sum_f \langle \psi_i | \hat{E}_{(x)}^{(-)} | \psi_f \rangle \langle \psi_f | \hat{E}_{(x)}^{(+)} | \psi_i \rangle \\ &= s \langle \psi_i | \hat{E}_{(x)}^{(-)} \underbrace{\sum_f (|\psi_f\rangle \langle \psi_f|)}_{\mathbb{1}} \hat{E}_{(x)}^{(+)} | \psi_i \rangle \\ &\quad \mathbb{1} \text{ since } \langle \psi_i | \hat{E}_{(x)}^{(+)} | \psi_i \rangle = 0 \end{aligned}$$

$$\Rightarrow W = s \langle \psi_i | \hat{E}_{(x)}^{(-)} \hat{E}_{(x)}^{(+)} | \psi_i \rangle$$

More generally, if the initial state is mixed  $\hat{\rho} = \sum p_i |\psi_i\rangle \langle \psi_i|$

$$\begin{aligned} \Rightarrow W &= s G^{(1)}(x, x) & G^{(1)}(x_1, x_2) &\equiv \langle \hat{E}_{(x_1)}^{(-)} \hat{E}_{(x_2)}^{(+)} \rangle \\ & & &= \text{Tr}(\hat{\rho} \hat{E}_{(x_1)}^{(-)} \hat{E}_{(x_2)}^{(+)}) \end{aligned}$$

Now consider n-fold coincidence counter



We seek the joint probability per unit (time)<sup>n</sup> that all n atoms are ionized

n<sup>th</sup> order perturbation theory! (see Glauber)

$$W^{(n)} = S^n G^{(n)}(x_1, x_2, x_3 \dots x_n; x_1, x_2, x_3 \dots x_n)$$

$$G^{(n)}(x_1, x_n, x'_1 \dots x'_n) = \langle \hat{E}^{(-)}(x_1) \dots \hat{E}^{(-)}(x_n) \hat{E}^{(+)}(x'_1) \dots \hat{E}^{(+)}(x'_n) \rangle$$

We have seen these correlation functions already in the the classical statistical theory. Glauber's formulation connects them explicitly to photon counting statistics for general quantum states of light.

Eg. HBT correlation

Classical  $G^{(2)}(t+\tau, t; t+\tau, t) = \langle E^*(t+\tau) E^*(t) E(t+\tau) E(t) \rangle$   
 $= \langle I(t+\tau) I(t) \rangle$

Quantum  $G^{(2)}(t+\tau, t; t+\tau, t) = \langle \hat{E}^{(-)}(t+\tau) \hat{E}^{(-)}(t) \hat{E}^{(+)}(t+\tau) \hat{E}^{(+)}(t) \rangle$

$= \langle : \hat{I}(t+\tau) \hat{I}(t) : \rangle$  normal order

## Properties of quantum correlation functions

Note: Since  $\rho$  is a positive operator

$$\text{Tr}(\rho \hat{A}^\dagger \hat{A}) \geq 0$$

$$\Rightarrow \text{with } \hat{A} = \hat{E}_{(x)}^{(+)} \quad G^{(1)}(x, x) \geq 0$$

$$\hat{A} = \hat{E}_{(x_1)}^{(+)} \cdots \hat{E}_{(x_n)}^{(+)} \quad G^{(n)}(x_1, \dots, x_n; x_1, \dots, x_n) \geq 0 \quad \left. \vphantom{\hat{A}} \right\} \text{Detection rates}$$

$$\hat{A} = \lambda_1 (\hat{E}_{(x_1)}^{(+)} \cdots \hat{E}_{(x_n)}^{(+)}) + \lambda_2 (\hat{E}_{(x'_1)}^{(+)} \cdots \hat{E}_{(x'_n)}^{(+)})$$

$$\Rightarrow \left[ \begin{array}{l} G^{(n)}(x_1, \dots, x_n; x_1, \dots, x_n) G^{(n)}(x'_1, \dots, x'_n; x'_1, \dots, x'_n) \\ \geq |G^{(n)}(x_1, \dots, x_n, x'_1, \dots, x'_n)|^2 \end{array} \right]$$

Note: This looks the same as the classical Cauchy-Schwarz inequality, but it is not because of non-commutativity of  $\hat{E}^{(+)}$  and  $\hat{E}^{(-)}$  (more soon)

## First order coherence for quantum fields

Back to Mach-Zehnder



$$I_{out} = \left[ G^{(1)}(t_1, t_1) + G^{(1)}(t_2, t_2) + 2 \operatorname{Re} \{ G^{(1)}(t_1, t_2) \} \right] / 4$$

$$= \frac{1}{4} \left[ G^{(1)}(t_1, t_2) + G^{(1)}(t_2, t_2) + 2 \sqrt{G^{(1)}(t_1, t_1) G^{(1)}(t_2, t_2)} |g^{(1)}(t_1, t_2)| \cos(\phi(t_1, t_2)) \right]$$

$$I_{stationary} = \frac{I_0}{2} (1 + |g^{(1)}(\tau)| \cos(\omega_0 \tau + \phi))$$

Same as before: Coherence  $|g^{(1)}(\tau)| = 1$

⇒ Factorization of correlation function

$$G^{(1)}(t_1, t_2) = E^*(t_1) E(t_2)$$

## Examples of states with first order coherence

• "Coherent state"  $|\{\alpha_k\}\rangle = \hat{D}(\{\alpha_k\})|0\rangle$

$$\hat{E}^{(+)}(x) |\{\alpha_k\}\rangle = E_c(x) |\{\alpha_k\}\rangle$$

$$\Rightarrow G^{(1)}(x_1, x_2) = E_c^*(x_1) E_c(x_2)$$

(Next Page)

• Single Photon state

$$G^{(1)}(x_1, x_2) = \langle 1 | \hat{E}^{(-)}(x_1) \hat{E}^{(+)}(x_2) | 1 \rangle$$

$$= \sum_{\psi} \langle 1 | \hat{E}^{(-)}(x_1) | \psi \rangle \langle \psi | \hat{E}^{(+)}(x_2) | 1 \rangle$$

Only Vacuum contributes  $= \langle 1 | \hat{E}^{(-)}(x_1) | 0 \rangle \langle 0 | \hat{E}^{(+)}(x_2) | 1 \rangle$

$$= \phi^*(x_1) \phi(x_2) \quad \leftarrow \text{"single photon wave function"}$$

$\Rightarrow$  Single photon state can exhibit same interference pattern in interferometer as classical light

• Single mode field in stationary state

$$\hat{\rho} = \sum_n P_n |n\rangle \langle n| \quad (\text{no coherences in } |n\rangle)$$

$$\hat{E}^{(+)}(\vec{x}, t) = \hat{a} \underbrace{E_{vac}(\vec{x}, t)}_{\rightarrow E_{vac}(\vec{x}, t)}$$

$$\Rightarrow G^{(1)}(x_1, x_2) = \langle \hat{a}^\dagger \hat{a} \rangle E_{vac}^*(x_1) E_{vac}(x_2) = \sum_n P_n n = \bar{n}$$

$$= (\sqrt{\bar{n}} E_{vac}^*(x_1)) (\sqrt{\bar{n}} E_{vac}(x_2))$$

## Higher-order correlation

Hanbury-Brown Twiss type~~s~~ photon counting

⇒ Correlation of two photons, arriving at beam splitter, separated by  $\tau$

$$G^{(2)}(\tau) \equiv \langle : \hat{I}(\tau) \hat{I}(0) : \rangle$$

$$= \langle \hat{E}^{(-)}(\tau) \hat{E}^{(-)}(0) \hat{E}^{(+)}(\tau) \hat{E}^{(+)}(0) \rangle$$

Normally ordered expectation values calculated via Glauber P-functions

$$\Rightarrow G^{(2)}(\tau) = \int d[\mathcal{E}] P[\mathcal{E}] |\mathcal{E}(\tau)|^2 |\mathcal{E}(0)|^2$$

If  $P[\mathcal{E}]$  is a positive normalizable function

⇒ State is a statistically mixture of coherent states

⇒  $P[\mathcal{E}]$  is a classical probability distr.

⇒ Integral is an inner product

$$G^{(2)}(\tau) = \langle I(\tau) | I(0) \rangle$$

(next page)

By the triangle inequality (Cauchy-Schwartz)

$$\underbrace{|\langle I(\tau) | I(0) \rangle|^2}_{|G^{(2)}(\tau)|^2} \leq \underbrace{\langle I(\tau) | I(\tau) \rangle}_{\langle \hat{I}(\tau) \hat{I}(\tau) \rangle} \underbrace{\langle I(0) | I(0) \rangle}_{\langle \hat{I}(0) \hat{I}(0) \rangle}$$

$\parallel$  Stationary  $\parallel$   
 $G^{(2)}(0)$   $G^{(2)}(0)$

$\Rightarrow$  Classical light  $G^{(2)}(\tau) \leq G^{(2)}(0)$

Normalized  $\Rightarrow$   $g^{(2)}(\tau) \leq g^{(2)}(0)$

Classical Light

- $\Rightarrow$  Rate of coincidence at zero delay
- $\geq$  Rate of coincidence at finite delay

Photon Bunching

If  $g^{(2)}(0) < g^{(2)}(\tau) \Rightarrow$  Photon more likely to arrive separately than together

$\Rightarrow$  Nonclassical light

Photon Antibunching

## Photon statistics

Consider single mode field. We saw that and stationary field showed first order coherence, classical or not.

$$\begin{aligned}
 g^{(2)}(0) &= \frac{\langle : \hat{n}^2 : \rangle}{\langle \hat{n} \rangle^2} = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} \\
 &= \frac{\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2}{\langle \hat{a}^\dagger \hat{a} \rangle^2} \\
 &= \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle^2} = \frac{\langle \Delta n^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2} + 1
 \end{aligned}$$

$$\Rightarrow g^{(2)}(0) - 1 = \frac{\langle \Delta n^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle^2}$$

Classically  $g^{(2)}(0) - 1 \geq 0$

Proof:  $g^{(2)}(0) = \frac{\int d^2\alpha P(\alpha) |\alpha|^4}{\langle n \rangle^2}$

$$\Rightarrow g^{(2)}(0) - 1 = \int d^2\alpha P(\alpha) (|\alpha|^4 - (\overline{|\alpha|^2})^2) \geq 0$$

if  $P(\alpha)$  is a probability

Thus, classically  $\langle \Delta n^2 \rangle \geq \langle n \rangle$

Poissonian or "Super Poissonian"

Quantum effect: Sub-Poissonian statistics

Comparison with Classical Mandel formula

$$P_n = \int d^2\alpha P(\alpha) e^{-|\alpha|^2/2} \frac{(|\alpha|^2)^n}{n!}$$

$$\Delta n^2 = \langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2$$

$$= \langle a^\dagger a a^\dagger a \rangle - \langle a^\dagger a \rangle^2$$

$$= \langle (a^\dagger)^2 (a)^2 \rangle + \langle a^\dagger a \rangle - \langle a^\dagger a \rangle^2$$

$$= \langle n \rangle + (\Delta \alpha^2)$$

Shot-noise

Fluctuation in classical Amplitude