

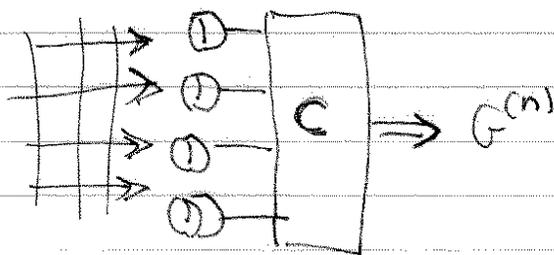
## Lecture 15: Squeezed States

So far we have explored classical vs. nonclassical light ~~by~~ by considering the correlation functions

$$G^{(n)}(x_1, x_2, \dots, x_n) = \langle \hat{I}(x_1) \hat{I}(x_2) \dots \hat{I}(x_n) \rangle$$

$$\hat{I}(x) = \hat{E}^{(-)}(x) \hat{E}^{(+)}(x)$$

This corresponds to a particular set of observables measured in a particular class of experiment:  
Coincidence counting in direct detection



These are not the only relevant observables. Another important quantum feature ~~to~~ to add to our list of photon-antibunching and sub-Poissonian statistics is squeezing.

This is a phase-dependent feature of quantum fluctuations that attracted tremendous attention in the 1980's and early 1990's due to the promise of reduced noise in communications and improvements in precision measurement. Unfortunately the promise was not realized but maybe it will be someday in the future.

# Phase Space Encore

Recall our picture of a single mode harmonic oscillator

Classical complex amplitude

$$\alpha = Q + iP$$

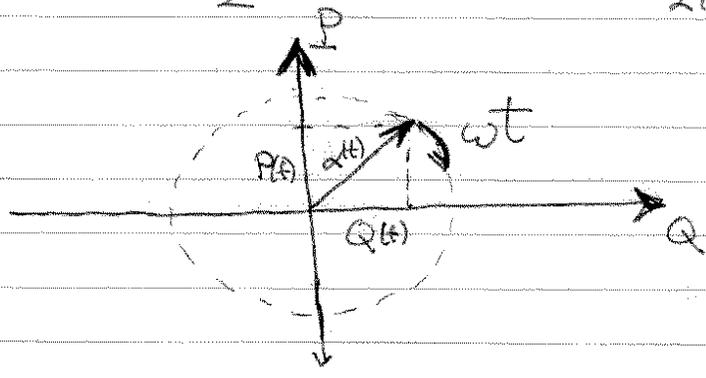
Classical solution:

$$\alpha(t) = \alpha(0) e^{-i\omega t} = A e^{i(\phi - \omega t)}$$

$$= \underbrace{[Q(0) \cos \omega t + P(0) \sin \omega t]}_{Q(t)} + i \underbrace{[P(0) \cos \omega t - Q(0) \sin \omega t]}_{P(t)}$$

$$Q = \frac{\alpha + \alpha^*}{2}$$

$$P = \frac{\alpha - \alpha^*}{2i}$$



Q and P are known as the "quadratures" are the oscillation; the parts oscillating in phase (i.e like cosine) and 90° out of phase (in quadrature; i.e. like sine)

For fields we sometime write

$$X_1 \equiv Q = \frac{\alpha + \alpha^*}{2}$$

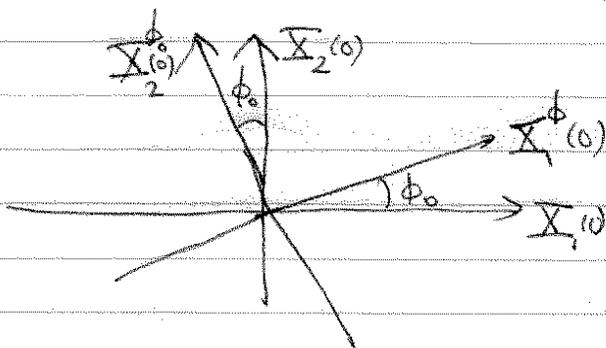
$$X_2 \equiv P = \frac{\alpha - \alpha^*}{2i}$$

More generally we define quadrature w.r.t. some phase  $\phi_0$

$$\underline{X}_1^{\phi_0} = \frac{\alpha e^{i\phi_0} + \alpha^* e^{-i\phi_0}}{2} \quad \underline{X}_2^{\phi_0} = \frac{\alpha e^{i\phi_0} - \alpha^* e^{-i\phi_0}}{2i}$$

$$\Rightarrow \alpha = (\underline{X}_1^{\phi_0} + i \underline{X}_2^{\phi_0}) e^{-i\phi_0}$$

$$\Rightarrow \text{Re}(\alpha(t) e^{-i\omega t}) = \underline{X}_1^{\phi_0}(t) \cos(\omega t + \phi_0) + \underline{X}_2^{\phi_0}(t) \sin(\omega t + \phi_0)$$



Quantum description

$$\alpha = \underline{X}_1 + i \underline{X}_2 \Rightarrow \hat{a} = \hat{X}_1 + i \hat{X}_2$$

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad [\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

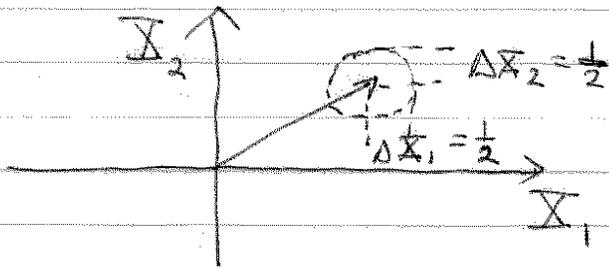
Uncertainty principle

$$\Delta \underline{X}_1 \Delta \underline{X}_2 \geq \frac{1}{4} \quad (\text{in these units})$$

More generally  $[\hat{X}_1^{\phi_0}, \hat{X}_2^{\phi_0}] = \frac{i}{2}$  for any quadratures

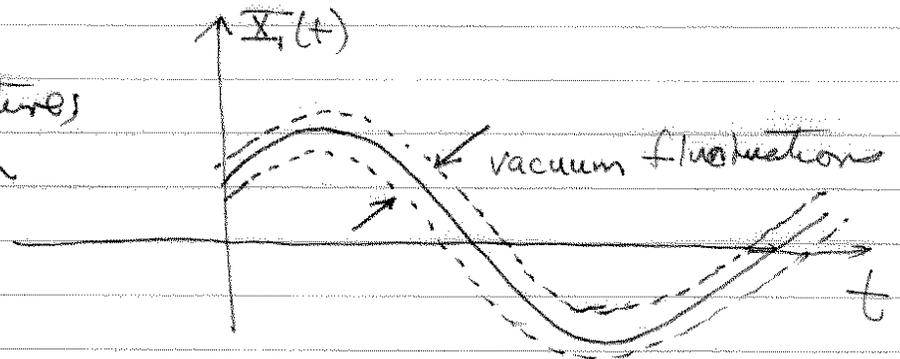
$$\Delta \underline{X}_1^{\phi_0} \Delta \underline{X}_2^{\phi_0} \geq \frac{1}{4}$$

Coherent state:  $\hat{a}|\alpha_0\rangle = \alpha_0|\alpha_0\rangle$



Minimum uncertainty state  $\Delta X_1^{\phi_0} = \Delta X_2^{\phi_0} = \frac{1}{2}$   
for any  $\phi_0$

Direct space pictures  
as a function  
of time



~~Big~~ Note: Wigner function for the  
coherent state

$$W(\alpha) = \frac{1}{\pi} e^{-2|\alpha - \alpha_0|^2} = \frac{1}{\sqrt{\pi}} e^{-2(X_1 - X_1^{(0)})^2} \frac{1}{\sqrt{\pi}} e^{-2(X_2 - X_2^{(0)})^2}$$

$$= W(X_1, X_2)$$

$\Delta X_1$  and  $\Delta X_2$

are the variances of the Wigner function

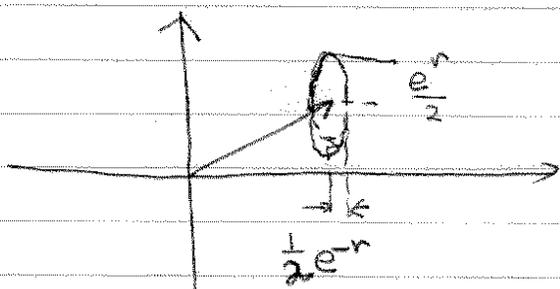
since  $W(X_1, X_2)$  yields the ~~also~~ correct  
marginal probability distributions  $P(X_1)$  and

$P(X_2)$ .  $\Rightarrow$  Error ellipse are the  
contours of the Wigner function at  
the rms value

## Squeezed State definition

Minimum uncertainty state of harmonic oscillator but with unequal variances in conjugate quadratures.

e.g.  $\Delta X_1 = \frac{e^{-r}}{2}$        $\Delta X_2 = \frac{e^{+r}}{2}$



"r" is known as the squeezing parameter

Such a state is "nonclassical" from the point of view of the P-function; i.e. a squeezed state cannot be thought of as a statistical mixture of coherent states

Proof:  $\Delta X_1^2 = \langle X_1^2 \rangle - \langle X_1 \rangle^2$   
 $= \frac{1}{4} (\langle (\hat{a} + \hat{a}^\dagger)^2 \rangle - \langle \hat{a} + \hat{a}^\dagger \rangle^2)$

{ Aside:  $\langle (\hat{a} + \hat{a}^\dagger)^2 \rangle = \langle \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} \rangle$   
 $= \langle :(\hat{a} + \hat{a}^\dagger)^2: \rangle + 1$   
 $= \int \frac{d^2\alpha}{\pi} P(\alpha) |\alpha + \alpha^*|^2 + 1 = \overline{|\alpha + \alpha^*|^2} + 1$

$\therefore \Delta X_1^2 = \frac{1}{4} (\overline{|\alpha + \alpha^*|^2} - (\overline{|\alpha + \alpha^*|})^2 + 1)$   
 $= \frac{1}{4} (\Delta |\alpha + \alpha^*|)^2 + \frac{1}{4} \geq \frac{1}{4}$       q.e.d.

Recall that a system with positive  $P(x) \Rightarrow$  source generated by current with classical stochastic fluctuations. These fluctuations can only increase  $\Delta X$  above the ~~the~~ quantum fluctuation. In this sense they are considered "nonclassical light". However as we will see, they are only marginally nonclassical, as Wigner function is positive.

Description of Squeezed State from Unitary Transformation

Consider  $\hat{S}(r) \equiv \exp\left\{\frac{1}{2}(r\hat{a}^{12} - r\hat{a}^{1+2})\right\}$

$-\hat{A} = -i\hat{H}$  anti-Hermitian

$\hat{S}^\dagger \hat{a} \hat{S}(r) = \hat{a} + [\hat{A}, \hat{a}] + \frac{1}{2}[\hat{A}, [\hat{A}, \hat{a}]] + \dots$

$[\hat{A}, \hat{a}] = -r\hat{a}^\dagger \quad [\hat{A}, [\hat{A}, \hat{a}]] = r^2\hat{a}$

$\Rightarrow \hat{S}^\dagger \hat{a} \hat{S} = \left(\sum_{n \text{ even}} \frac{r^n}{n!}\right) \hat{a} - \left(\sum_{n \text{ odd}} \frac{r^n}{n!}\right) \hat{a}^\dagger$

$\boxed{\hat{S}^\dagger \hat{a} \hat{S} = \cosh(r)\hat{a} - \sinh(r)\hat{a}^\dagger}$

Bogoliubov transformation

Like rotation  $a \rightarrow a^\dagger$  by imaginary angle  
(Related to Lorentz transformation)

Transformation of  $\hat{X}_1$  and  $\hat{X}_2$

$$\hat{S}^{\dagger} \hat{X}_1 \hat{S} = \frac{1}{2} (\hat{S}^{\dagger} \hat{a} \hat{S} + \hat{S}^{\dagger} \hat{a}^{\dagger} \hat{S})$$

$$= \frac{1}{2} (c \hat{a} - s \hat{a}^{\dagger} + c \hat{a}^{\dagger} - s \hat{a})$$

$$c \equiv \cosh r$$

$$s \equiv \sinh r$$

$$= (c-s) \left( \frac{\hat{a} + \hat{a}^{\dagger}}{2} \right) = e^{-r} \hat{X}_1$$

$$\hat{S}^{\dagger} \hat{X}_2 \hat{S} = e^{+r} \hat{X}_2$$

Note: (the Bogoliubov transformation linearly related  $\hat{a}$  and  $\hat{a}^{\dagger}$  and  $\hat{X}_1, \hat{X}_2$  because the generator is quadratic in  $\hat{a}$  and  $\hat{a}^{\dagger}$ )

"Squeezed vacuum"

$$|0_r\rangle \equiv \hat{S}(r)|0\rangle$$

$$\Rightarrow \langle 0_r | \hat{a} | 0_r \rangle = \langle 0_r | \hat{a}^{\dagger} | 0_r \rangle = 0$$

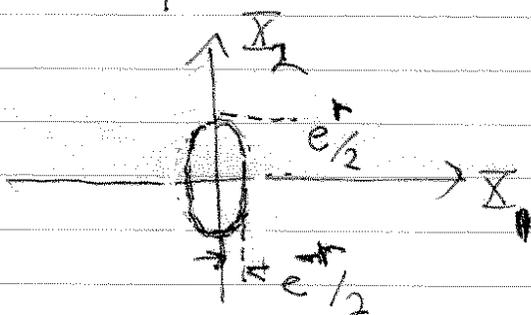
$$\langle \Delta \hat{X}_1^2 \rangle_r = \langle \hat{X}_1^2 \rangle_r + \langle \hat{X}_1 \rangle_r^2 \xrightarrow{0} \langle 0_r | \hat{X}_1^2 | 0_r \rangle$$

$$= \langle 0 | (\hat{S}^{\dagger} \hat{X}_1 \hat{S})^2 | 0 \rangle = e^{-2r} \langle 0 | \hat{X}_1^2 | 0 \rangle$$

$$\langle \Delta \hat{X}_1^2 \rangle_r = \frac{e^{-2r}}{4}$$

Similarly

$$\langle \Delta \hat{X}_2^2 \rangle_r = \frac{e^{+2r}}{4}$$



Note:  $\langle \hat{n} \rangle_r = \langle 0_r | \hat{a}^\dagger \hat{a} | 0_r \rangle = \langle 0 | \hat{a}_r^\dagger \hat{a}_r | 0 \rangle$

$$= \langle 0 | (e\hat{a} + s\hat{a}^\dagger)^\dagger (e\hat{a} + s\hat{a}^\dagger) | 0 \rangle$$

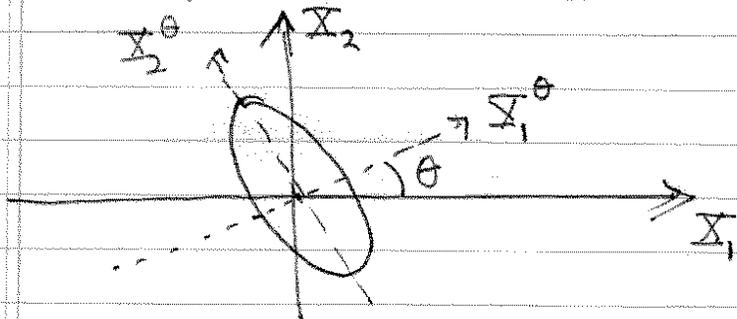
$$= \sinh^2 r \neq 0$$

$\Rightarrow$  Squeezed "vacuum" has photons

$$|0_r\rangle = \hat{S}(r) |0\rangle = e^{r(\hat{a}^2 - \hat{a}^{\dagger 2})} |0\rangle$$

$\Rightarrow$  Only even photon numbers

More general case



Squeezed vacuum  
w.r.t different  
quadratures

$$\hat{X}_1^\theta = \hat{R}^\dagger(\theta) \hat{X}_1 \hat{R}(\theta) \quad \hat{X}_2^\theta = \hat{R}^\dagger(\theta) \hat{X}_2 \hat{R}(\theta)$$

$$\hat{R}(\theta) = e^{-i\theta \hat{a}^\dagger \hat{a}} \quad \hat{R}^\dagger(\theta) \hat{a} \hat{R}(\theta) = \hat{a} e^{-i\theta}$$

$$\Rightarrow \hat{R}^\dagger(\theta) \hat{S}(r) \hat{R}(\theta) = \exp \left\{ \frac{r}{2} [(\hat{R}^\dagger \hat{a} \hat{R})^2 - (\hat{R}^\dagger \hat{a}^\dagger \hat{R})^2] \right\}$$

$$= \exp \left\{ \frac{1}{2} (r e^{-i2\theta} \hat{a}^2 - r e^{i2\theta} \hat{a}^{\dagger 2}) \right\}$$

$$\Rightarrow \hat{S}(\zeta) = \exp \left\{ \frac{1}{2} (\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2}) \right\}$$

$\zeta = r e^{2i\theta}$  : Squeezing parameter (complex)

### General Bogolubov Transformation

$$\begin{aligned}\hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) &= \hat{R}^\dagger(\theta) \hat{S}^\dagger(r) \underbrace{\hat{R}(\theta) \hat{a} \hat{R}^\dagger(\theta)}_{e^{i\theta} \hat{a}} \hat{S}(r) \hat{R}^\dagger(\theta) \\ &= e^{i\theta} \hat{R}^\dagger(\theta) (\cosh(r) \hat{a} - \sinh(r) \hat{a}^\dagger) \hat{R}(\theta)\end{aligned}$$

$$\Rightarrow \begin{cases} \hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) = \cosh(r) \hat{a} - e^{2i\theta} \sinh(r) \hat{a}^\dagger \\ \hat{S}^\dagger(\zeta) \hat{a}^\dagger \hat{S}(\zeta) = \cosh(r) \hat{a}^\dagger - e^{-2i\theta} \sinh(r) \hat{a} \end{cases}$$

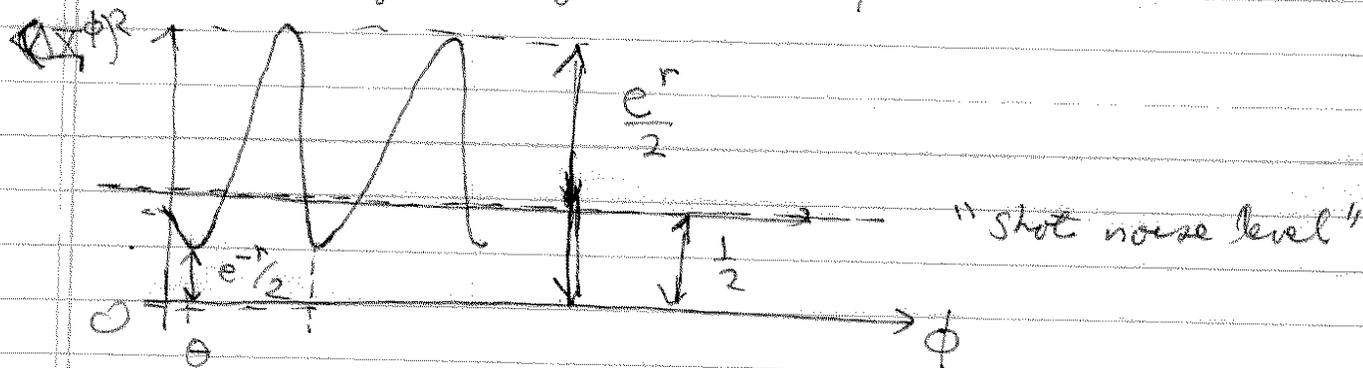
Fluctuations in  $\hat{X}_1^\phi$  and  $\hat{X}_2^\phi$

$$\begin{aligned}\langle (\Delta X_1^\phi)^2 \rangle_\zeta &= \langle (\hat{X}_1^\phi)^2 \rangle_\zeta - \langle \hat{X}_1^\phi \rangle_\zeta^2 = \frac{1}{4} \langle (\hat{R}^\dagger(\theta) \hat{X}_1 \hat{R}(\theta))^2 \rangle_\zeta \\ &= \frac{1}{4} \langle 0_\zeta | (\hat{a} e^{-i\phi} + \hat{a}^\dagger e^{i\phi})^2 | 0_\zeta \rangle \\ &= \frac{1}{4} \langle 0_\zeta | [ (e e^{-i\phi} - e^{-2i\theta} e^{i\phi} \lambda) \hat{a} + (e e^{i\phi} + e^{2i\theta} e^{-i\phi} \lambda) \hat{a}^\dagger ]^2 | 0_\zeta \rangle \\ &= \frac{1}{4} | e e^{-i\phi} - e^{-2i\theta + i\phi} \lambda |^2 = \frac{1}{4} | e - e^{-2i(\theta - \phi)} \lambda |^2 \\ &= \frac{1}{4} (e^2 + \lambda^2 - 2 \cos 2(\theta - \phi) \lambda e)\end{aligned}$$

$$\Rightarrow \langle (\Delta X_1^\phi)^2 \rangle_\zeta = \frac{1}{4} [ e^{-2r} \cos^2(\theta - \phi) + e^{+2r} \sin^2(\theta - \phi) ]$$

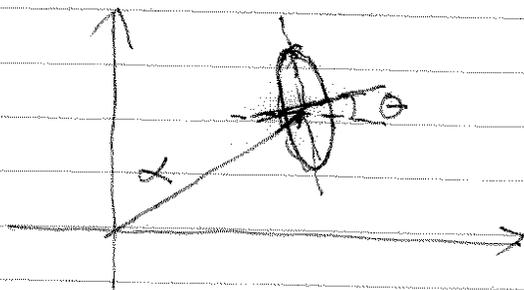
Similarly  $\langle (\Delta X_2^\phi)^2 \rangle_\zeta = \frac{1}{4} [ e^{+2r} \cos^2(\theta - \phi) + e^{-2r} \sin^2(\theta - \phi) ]$

Thus the squeezed vacuum state shows periodic variation in the fluctuation as a function of the quadrature  $\phi$



Squeezed coherent state:

$$|\alpha, \zeta\rangle \equiv \hat{D}(\alpha) |0_\zeta\rangle = \hat{D}(\alpha) \hat{S}(\zeta) |0\rangle$$



Fluctuations in quadrature same as  $|0_\zeta\rangle$  but with mean values

$$\langle \alpha, \zeta | \hat{X}_1 | \alpha, \zeta \rangle = \text{Re}(\alpha)$$

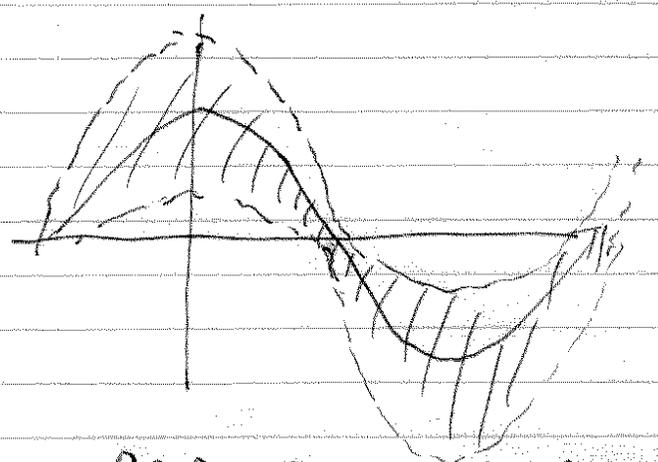
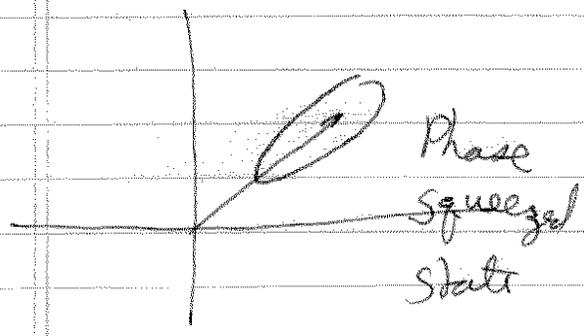
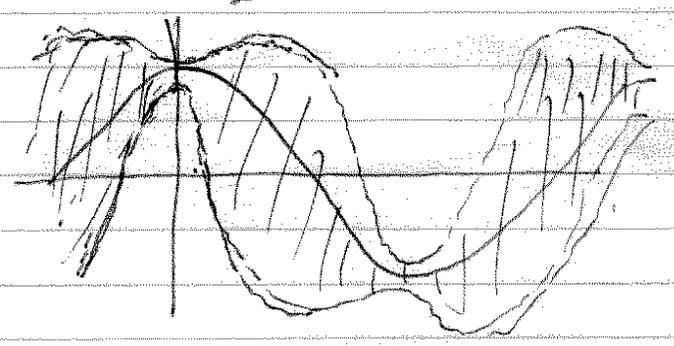
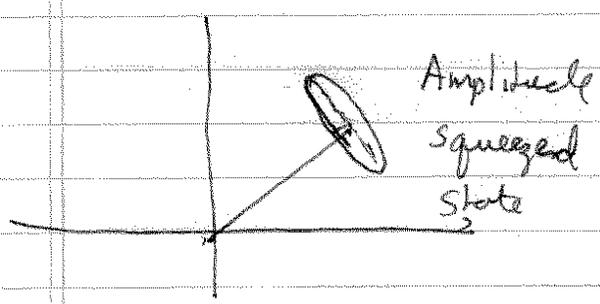
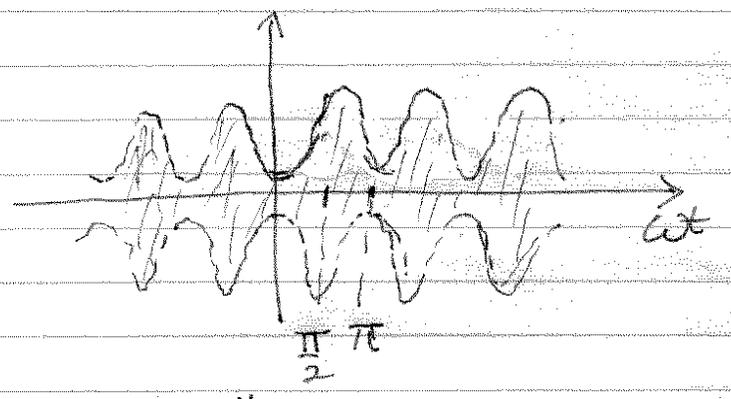
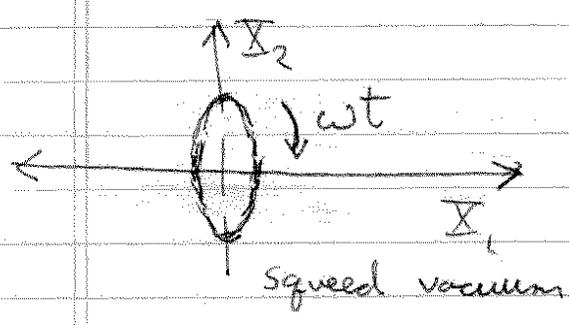
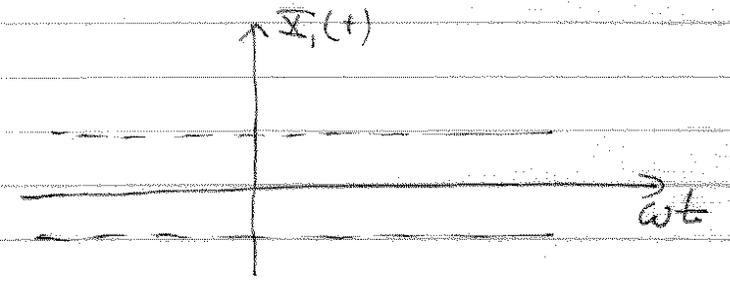
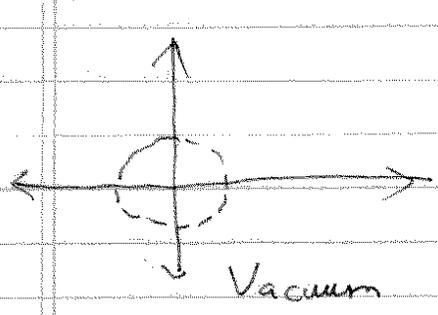
$$\langle \alpha, \zeta | \hat{X}_2 | \alpha, \zeta \rangle = \text{Im}(\alpha)$$

$$\langle \Delta \hat{X}_1^2 \rangle \text{ and } \langle \Delta \hat{X}_2^2 \rangle \text{ as for } |0_\zeta\rangle$$

Note:  $\hat{D}(\alpha) \hat{S}(\zeta) \neq \hat{S}(\zeta) \hat{D}(\alpha)$

We define things by squeezing first, then displacing.

From phase space to time evolution



See CM Caves, PRD 23, 1693 (1981).