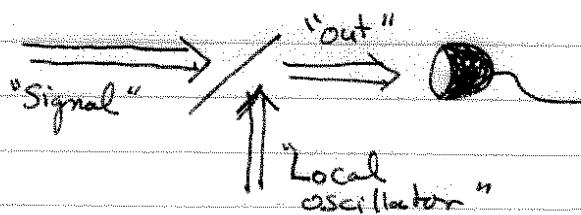


## Lecture 16: Squeezed States: Production and Detection

### Homodyne detection:

We saw last lecture squeezed states which have reduced fluctuations along some phase quadratures and increased fluctuations along others. In order to detect these phase sensitive noise properties, we need a phase sensitive detection system. The basic idea is to use interference.



Classically, the signal is  $\text{Re}(\mathcal{E}_m e^{-i\omega t})$ . The local oscillator is a strong, well defined field oscillating at  $\omega$   $\text{Re}(\mathcal{E}_L e^{-i\omega_L t})$ . The "out" field is then the beat note. If  $\omega = \omega_L$  this ~~det~~ detection is known as "homodyne" otherwise "heterodyne". We will consider homodyne first.

The output complex amplitude is

$$\mathcal{E}_{\text{out}} = t \mathcal{E}_m + r \mathcal{E}_L \quad \text{where } t = \text{transmission coeff.} \\ r = \text{reflection coeff.}$$

$$\Rightarrow I_{\text{out}} = |\mathcal{E}_{\text{out}}|^2 = |t|^2 I_m + |r|^2 I_L$$

$$+ (t^* r \mathcal{E}_m^* \mathcal{E}_L + t r^* \mathcal{E}_m \mathcal{E}_L^*)$$

Now for a symmetric beam splitter  $t^*r = i|t| |r|$

$$\Rightarrow I_{\text{out}} = |t|^2 I_{\text{in}} + |r|^2 I_L$$

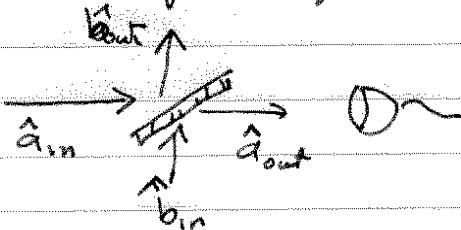
$$+ |t| |r| |\varepsilon_L| (\underbrace{\varepsilon_{\text{in}}^* e^{i(\phi_L + \frac{\pi}{2})} + \varepsilon_{\text{in}} e^{-i(\phi_L + \frac{\pi}{2})}}$$

$$\rightarrow 2 \sum_{\text{in}} (\phi_L + \frac{\pi}{2})$$

Quadrature in input determined by the phase of the local oscillator

The homodyne signal thus consists the transmitted input, the reflected local oscillator, and the interference term which measures some phase quadrature of the input depending on the phase of the local oscillator.

Let us now quantify the fields



$$\hat{a}_{\text{out}}^{\dagger} \hat{a}_{\text{out}} = |t|^2 \hat{a}_{\text{in}}^{\dagger} \hat{a}_{\text{in}} + |r|^2 \hat{b}_{\text{in}}^{\dagger} \hat{b}_{\text{in}} + t^* r \hat{a}_m^{\dagger} \hat{b}_{\text{in}} + r^* t \hat{a}_{\text{in}}^{\dagger} \hat{b}_m +$$

As we saw in P.S. #5, the linear transformation are the same as for the classical field

Take the input to port-b as a coherent state

$$|\beta_L\rangle \quad \beta_L = |\beta_L| e^{i\phi_L}$$

$$\hat{a}_{\text{out}}^{\dagger} = \hat{a}_{\text{out}}^{\dagger} \hat{a}_{\text{out}} = |t|^2 \langle \hat{a}_{\text{in}} \rangle + |r|^2 |\beta_L|^2 + 2|t| |r| |\beta_L| \langle \hat{X}_{\text{in}} (\phi_L + \frac{\pi}{2}) \rangle$$

$$\hat{X}_{\text{in}} (\phi) = \hat{a}_{\text{in}} e^{-i\phi} + \hat{a}_{\text{in}}^{\dagger} e^{i\phi}$$

Ordinary homodyne: We assume the input is a weak signal. In order not to degrade the signal/noise we take the beam splitter to be almost perfectly transmitting  
 $\Rightarrow |t| \gg |r|$

Furthermore, we take the local oscillator to have an overcompensating large intensity so that

$$|r|^2 |\beta_L|^2 \gg |t|^2 \langle \hat{n}_m \rangle$$

$$\Rightarrow \langle \hat{n}_{\text{out}} \rangle = \underbrace{|r|^2 |\beta_L|^2}_{\text{fixed background}} + \underbrace{2|t| |r| \beta_L K \hat{X}_{\text{in}} (\phi + \frac{\pi}{2})}_{\text{important signal}}$$

We are concerned with noise properties which appear in the fluctuations.

$$\langle \Delta \hat{n}_{\text{out}}^2 \rangle = \langle \hat{n}_{\text{out}}^2 \rangle - \langle \hat{n}_{\text{out}} \rangle^2$$

$$\begin{aligned} \text{At sides } \langle \hat{n}_{\text{out}}^2 \rangle &\approx R^2 \langle b_m^\dagger b_m^\dagger b_{in}^\dagger b_{in} \rangle + 4R\sqrt{TR} \\ &\quad \langle \hat{n}_{in}^b (a_m^\dagger b_m^\dagger - i b_m^\dagger a_m) \rangle \\ &\quad + (a_m^\dagger b_m^\dagger - i b_m^\dagger a_m) A_m^b \\ &\quad + TR \langle (a_m^\dagger b_m^\dagger - i a_m^\dagger b_m^\dagger)^2 \rangle \\ &= R^2 (|\beta_L|^4 + |\beta_L|^2) + 4R\sqrt{TR} |\beta_L|^3 \langle \hat{X}_{\text{in}} (\phi + \frac{\pi}{2}) \rangle \\ &\quad + 4TR \langle \hat{X}_{\text{in}}^2 (\phi + \frac{\pi}{2}) \rangle |\beta_L|^2 \end{aligned}$$

$$\Rightarrow \langle \Delta \hat{n}_{\text{out}}^2 \rangle \approx R |\beta_L|^2 (R + 4T \langle \Delta \hat{X}_{\text{in}}^2 (\phi + \frac{\pi}{2}) \rangle)$$

↑                              ↑

Local oscillator                      Input quadrature

Shot noise                      fluctuations

The ordinary homodyne fluctuation has two components

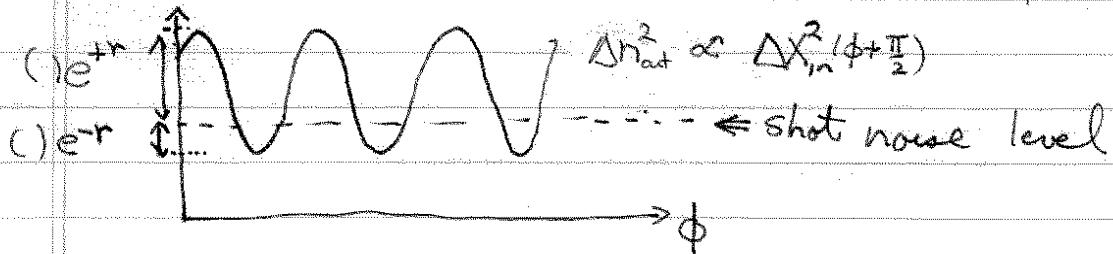
- (1) Reflected shot noise from the local oscillator
- (2) transmitted quadrature fluctuations from the input

Under the assumption  $R \ll T$  the shot noise contribution from the L.O. is negligible

$$\Rightarrow \langle \Delta \hat{n}_{\text{out}}^2 \rangle \approx 4TR|\beta_2|^2 \langle \Delta \hat{x}_{\text{in}}^2(\phi + \frac{\pi}{2}) \rangle$$

If the input is squeezed then for some values of the local oscillator phase  $\langle \Delta \hat{x}_{\text{in}}^2(\phi + \frac{\pi}{2}) \rangle < \frac{1}{4}$

$\Rightarrow$  Subshot-noise fluctuation for some  $\phi$

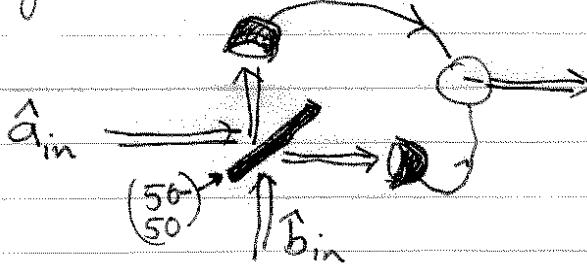


Operationally, the shot noise level is determined by blocking the input signal in which case the input is the vacuum.

Note: Ordinary homodyne detection relies on certain approximations for its efficacy. Most importantly, we must have minimal contribution to  $\Delta n_{\text{out}}^2$  from reflected L.O. noise. This required  $R \ll T$  and very low L.O. noise. Here we have assumed the minimal possible L.O. noise as a coherent state. Other technical noise in the L.O. will contaminate  $\Delta n_{\text{out}}^2$ .

## Balanced homodyne

We can remove some of the limitations of ordinary homodyne detection in a "balanced" geometry



The output is detected from both ports and then subtracted. The beam splitter is 50-50

$$\hat{a}_{out} = \frac{\hat{a}_{in} + i\hat{b}_{in}}{\sqrt{2}}$$

$$\hat{b}_{out} = \frac{\hat{b}_{in} + i\hat{a}_{in}}{\sqrt{2}}$$

$$\hat{n}_{out}^a = \hat{a}_{out}^\dagger \hat{a}_{out} = \frac{1}{2} (\hat{n}_{in}^a + \hat{n}_{in}^b + i(\hat{a}_{in}^\dagger \hat{b}_{in} - \hat{a}_{in} \hat{b}_{in}^\dagger))$$

$$\hat{n}_{out}^b = \hat{b}_{out}^\dagger \hat{b}_{out} = \frac{1}{2} (\hat{n}_{in}^a + \hat{n}_{in}^b + i(\hat{b}_{in}^\dagger \hat{a}_{in} - \hat{b}_{in} \hat{a}_{in}^\dagger))$$

$$\Rightarrow \hat{n}_{out} = \hat{n}_{out}^a - \hat{n}_{out}^b = i(\hat{a}_{in}^\dagger \hat{b}_{in} - \hat{a}_{in} \hat{b}_{in}^\dagger)$$

Assuming no correlation between  $a_{in}$  and  $b_{in}$

$$\langle \hat{n}_{out} \rangle = i(\beta_L \hat{a}_{in}^\dagger - \beta_L^* \hat{a}_{in}) = 2|\beta_L| \langle \hat{X}_{in}(\phi + \frac{\pi}{2}) \rangle$$

where  $\beta_L = \langle \hat{b}_{in} \rangle$  (not necessarily perfect coherent state)

$$\text{Then } \langle \hat{n}_{out}^2 \rangle = \langle (\hat{n}_{out}^a)^2 \rangle - \langle (\hat{n}_{out}^b)^2 \rangle$$

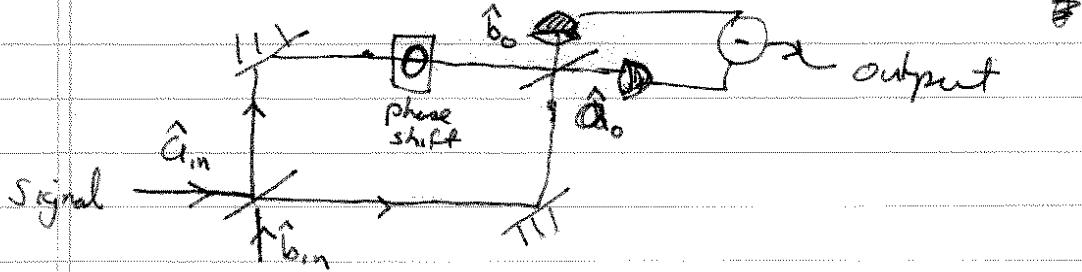
$$\Rightarrow \boxed{\langle \hat{n}_{out}^2 \rangle = 4|\beta_L|^2 \langle \hat{X}_{in}^2(\phi + \frac{\pi}{2}) \rangle}$$

Noise in the Local Oscillator is subtracted out. The only remaining noise is quadrature fluctuation in the input signal

## Squeezed States in precision interferometry

The highest precision measurement are accomplished using interferometry. An important and historic example for quantum optics is gravity wave detection (LIGO)

Consider a Mach Zender interferometer



$$\hat{a}_o = e^{i\theta/2} (\cos \frac{\theta}{2} \hat{a}_{in} + \sin \frac{\theta}{2} \hat{b}_{in})$$

$$\hat{b}_o = e^{i\theta/2} (\cos \frac{\theta}{2} \hat{b}_{in} + \sin \frac{\theta}{2} \hat{a}_{in})$$

$$\Rightarrow \hat{n}_o = \hat{a}_o^\dagger \hat{a}_o - \hat{b}_o^\dagger \hat{b}_o = \cos \theta (\hat{a}_i^\dagger \hat{a}_i - \hat{b}_i^\dagger \hat{b}_i) - i \sin \theta (\hat{a}_i^\dagger \hat{b}_i^\dagger - \hat{a}_i^\dagger \hat{b}_i)$$

If no signal is input into port - b  $\langle \hat{b}_i \rangle = \langle \hat{b}_i^\dagger \rangle = 0$

$$\langle \hat{n}_o \rangle = \cos \theta (\langle \hat{n}_i^a \rangle - \langle \hat{n}_i^b \rangle)$$

$$\langle \langle \hat{n}_o^2 \rangle \rangle = \cos^2 \theta (\langle \langle \hat{n}_i^a \rangle \rangle^2 + \langle \langle \hat{n}_i^b \rangle \rangle^2)$$

$$+ \sin^2 \theta (\langle \hat{n}_i^a \rangle + \langle \hat{n}_i^b \rangle + 2 \langle \hat{n}_i^a \rangle \langle \hat{n}_i^b \rangle)$$

$$+ \langle \hat{a}_i^2 \rangle \langle \hat{b}_i^{\dagger 2} \rangle + \langle \hat{a}_i^{\dagger 2} \rangle \langle \hat{b}_i^2 \rangle)$$

Suppose nothing enters port-b. There is of course vacuum entering

$$\Rightarrow \langle \hat{n}_0 \rangle = \cos \theta \langle \hat{n}_i^a \rangle$$

$$\langle \Delta \hat{n}_0^2 \rangle = \cos^2 \theta \langle (\Delta \hat{n}_i^a)^2 \rangle + \sin^2 \theta \langle \hat{n}_i^a \rangle$$

The second term is a purely quantum term arising from the beating of the ~~vacuum~~ fluctuations entering port-b with the signal in port-a

If we seek to measure changes in phase  $\delta\theta$  (as in a gravity wave detector) we should operate at the point of greatest sensitivity where the derivative of the signal is the greatest

$$\Rightarrow \text{Choose } \theta = \frac{\pi}{2} + \delta\theta$$

$$\therefore \langle \hat{n}_0 \rangle = \sin \delta\theta \langle \hat{n}_i^a \rangle$$

$$\langle \Delta \hat{n}_0^2 \rangle = \langle \hat{n}_i^a \rangle \text{ (shot noise)} \quad \text{to lowest order in } \delta\theta$$

Signal to noise ratio

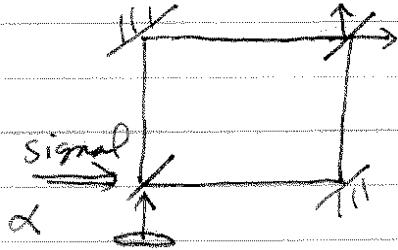
$$\text{SNR} = \frac{\langle \hat{n}_0 \rangle}{\sqrt{\Delta n_0^2}} = \sin \delta\theta \sqrt{\langle \hat{n}_i^a \rangle}$$

Minimal detectable phase shift

$$\delta\theta_{\min} \approx \frac{1}{\sqrt{\langle \hat{n}_i^a \rangle}}$$

The SNR is limited by the power of the input signal and detector integration time

Suppose now we inject "squeezed vacuum" into the unused port (Caves)



The reduced shot noise in port - b can improve the signal to noise, everything else held fixed.

Choosing squeezed vacuum of the form  $|n_b\rangle = e^{\frac{r}{2}(a-a^\dagger)^2} |0\rangle_b$

$$\langle \hat{n}_b \rangle = \sinh^2 r \quad \langle \Delta \hat{n}_b^2 \rangle = (\cosh 2r)(\sinh^2 r)$$

$$\begin{aligned} \langle \hat{b}_b^2 \rangle &= \langle (\hat{x}_1 + i\hat{x}_2)^2 \rangle = \langle \hat{x}_1^2 \rangle - \langle \hat{x}_2^2 \rangle \\ &= \langle \hat{b}_1^{+2} \rangle = \frac{1}{4} (e^{-2r} - e^{2r}) = -\frac{\sinh 2r}{2} \end{aligned}$$

$$\Rightarrow \langle \hat{n}_b \rangle = \cos \theta (|\alpha|^2 - \sinh^2 r)$$

$$\langle \Delta \hat{n}_b^2 \rangle = \cos \theta (|\alpha|^2 + (\sinh^2 r) \cosh 2r)$$

$$+ \sin^2 \theta (|\alpha|^2 (1 - 2 \sinh^2 r) - \frac{1}{2} (\alpha^2 + \alpha^{+2}) \sinh 2r + \sinh^2 r)$$

for  $|\alpha|^2 \gg \sinh^2 r$  (large signal)

With  $\theta = \pi_2 + \delta \theta$

~~$$\text{Ansatz } \langle \hat{n}_b \rangle \approx \sin \delta \theta |\alpha|^2$$~~

$$\langle \Delta \hat{n}_b^2 \rangle \approx |\alpha|^2 e^{-r}$$

$$\Rightarrow \boxed{\text{SNR} = \sqrt{\langle \hat{n}_b \rangle} e^r \sin \delta \theta}$$

$$\delta \theta_{\min} = e^{-r} \langle \hat{n}_b \rangle^{-1/2}$$

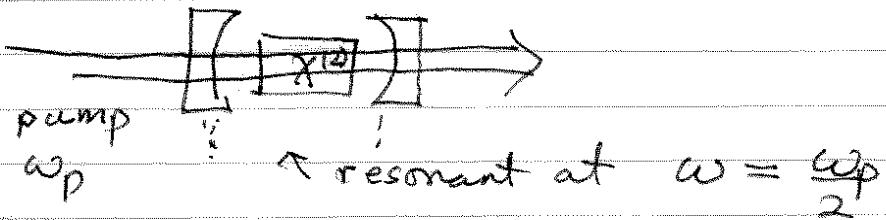
Improved  
Sensitivity

## Production of Squeezed state

Virtually any nonlinear optical phenomenon

(Second harmonic generation, parametric amplification, four wave mixing, the Kerr effect)

will lead to squeezing as the lowest order quantum optical effect. Though the first demonstration of squeezing was via 4-wave mixing in an atomic gas, today the best squeezing is achieved via parametric oscillation



The Hamiltonian for the cavity mode

$$\hat{H} = \chi^{(2)} \frac{|E_p| E_{\text{vac}}^2}{2} (\hat{a}^2 e^{i\phi_p} + \hat{a}^\dagger \hat{a}^2 e^{i\phi_p})$$

$$E_{\text{vac}} = \sqrt{\frac{2\pi\hbar\omega}{V}} \quad \text{cavity mode volume}$$

$$\Rightarrow \hat{U}(t) = e^{\frac{i}{2}\hat{a}^2 - \frac{i}{2}\hat{a}^\dagger \hat{a}^2}$$

$$\text{where } \gamma = \frac{i\chi^{(2)} |E_p| E_{\text{vac}}^2}{\hbar} t e^{i\phi_p}$$

The degenerate parametric oscillator produces squeezed vacuum below oscillation threshold. The squeezed quadrature depends on the pump phase

We saw the  $\hat{U}$  gives the Bogoliubov transformation  
it gives squeezing. We can also see this in  
the Heisenberg picture.

Choose  $\phi_p = 0$ , and let  $\hbar G = |\mathcal{E}_p| \mathcal{E}_{\text{vac}}^2 \chi^{(2)}$

$$\hat{H} = \frac{G}{2} (\hat{a}^2 + \hat{a}^{+2})$$

$$\frac{d\hat{a}}{dt} = -\frac{i}{\hbar} [\hat{a}, \hat{H}] = -iG\hat{a}^+$$

$$\frac{d\hat{a}^+}{dt} = +iG\hat{a}$$

$$\Rightarrow \frac{d^2\hat{a}}{dt^2} = G\hat{a} \Rightarrow \frac{d^2\hat{a}}{dt^2} - G\hat{a} = 0$$

$$\Rightarrow \hat{a}(t) = \cosh(Gt) \hat{a}(0) + \frac{1}{G} \sinh(Gt) \frac{d\hat{a}(0)}{dt}$$

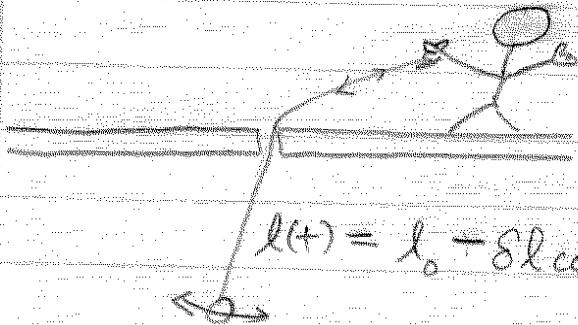
$$\boxed{\hat{a}(t) = \cosh(Gt) \hat{a}(0) + i \sinh(Gt) \hat{a}^+(0)}$$

Bogoliubov transformation with  $\xi = iG$

This linear equation is familiar in  
dynamics  $\Rightarrow$  parametric amplification

The prototypical example is a child on a  
swing. By pumping at twice the  
resonance frequency

## Classical Parametric Amplifier



$$l(t) = l_0 - \delta l \cos(\omega_p t + \phi_p)$$

Driving force which periodically changes the length of the pendulum

$$\omega^2(t) = \frac{g}{l(t)} = \frac{g}{l_0 - \delta l \cos \omega_p t} \approx \omega_0^2 + \epsilon \cos(\omega_p t + \phi_p)$$

$$\epsilon = \frac{\delta l}{l_0} \omega_0^2$$

Equation of motion:  $\ddot{x} + \omega^2(t)x = 0$

$$\Rightarrow \boxed{\ddot{x} + \omega_0^2 x + (\epsilon \cos(\omega_p t + \phi_p))x = 0}$$

Mattieu eqn.

Generally no solution, not integrable

For weak driving, can approximate

$$\begin{aligned} x(t) &= Q(t) \cos \omega t + P(t) \sin \omega t \\ &= R e(\alpha(t) e^{-i \omega t}) \end{aligned}$$

Take  $|\alpha| \ll \omega_0 |\alpha|$

Use complex amplitude (Take real part in end)

→ Eqn of motion:

$$(-\omega_0^2 \alpha - i\omega_0 \dot{\alpha} + \ddot{\alpha}) e^{-i\omega t} + \omega_0^2 \alpha e^{-i\omega t}$$

$$+ \epsilon e^{-i(\phi t + \phi)} \left( \alpha e^{-i\omega t} + \alpha^* e^{+i\omega t} \right) / 2$$

$$\text{with } |\dot{\alpha}| \ll \omega_0 \dot{\alpha}$$

$$\rightarrow \dot{\alpha} = e^{-i(\phi + \frac{\pi}{2})} \frac{\epsilon}{2\omega_0} (a e^{-i\omega t} + a^* e^{-(\omega_0 - 2\omega)t})$$

Parametric resonance

$$\omega_p = 2\omega_0$$

Ignore oscillating term (small correction)

→

$$\dot{\alpha} = -i\omega r e^{i\phi} \alpha^*$$

$$r = \frac{\epsilon}{2\omega_0}$$

Linear eqn

Special case  $\phi_p = -\frac{\pi}{2}$

→

$$\dot{\alpha} = -r \alpha^* \quad (r = \frac{\epsilon}{2\omega_0} \text{ real})$$

$$\dot{Q} = +Q$$

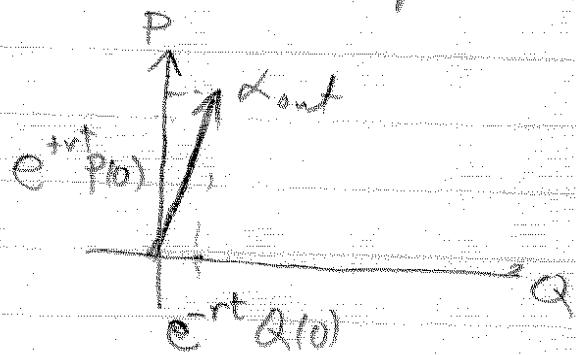
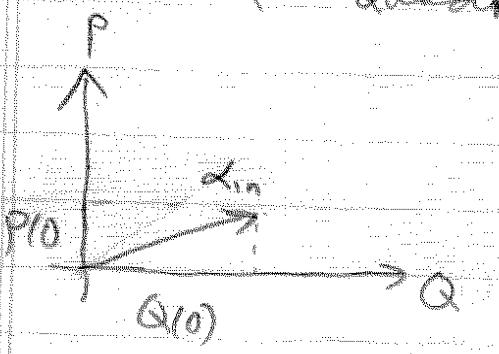
$$\dot{P} = +r P$$

$$\rightarrow Q(t) = e^{-rt} Q(0), \quad P(t) = e^{+rt} P(0)$$

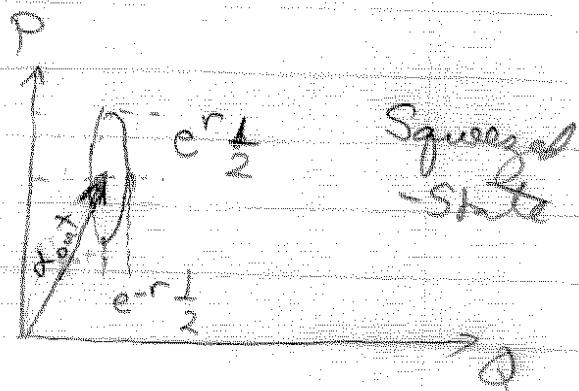
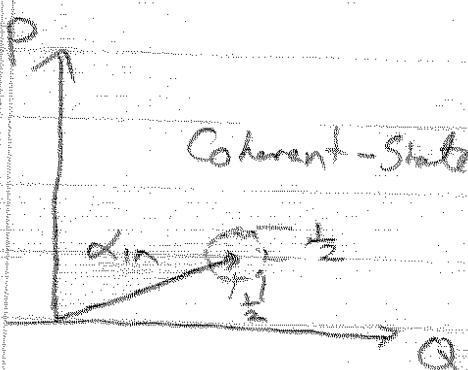
Phase sensitive amplification

P Quadrature amplified

Q Quadrature de-amplified



Given a distribution of amplitudes (uncertain "bubble") these are amplified in same phase sensitive manner



Arbitrary pump phase phase

$\rightarrow$  Complex Squeezing parameter

$$\xi = r^{2i\theta}$$

$$2\theta = \left(\frac{\pi}{2} + \phi_p\right)$$