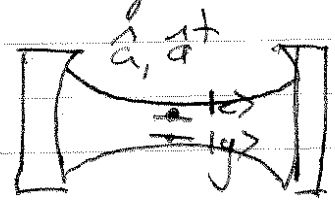
Lecture 17: The Jaynes-Cummings Problem

Besides the spontaneous emission problem we have considered two kinds of semiclassical problems

- Quantum atom interacting with classical field (e.g. Rabi flopping)
- Quantum field interacting with classical matter (e.g. Parametric downconversion)

As we know the world to be quantum mechanical at its foundation, we would like to see these theories as limits to a more fundamental model. We thus begin our exploration of coupled quantum systems. This investigation will bring to the very important problem of <u>open quantum systems</u> and its relation to dissipation, decoherence, irreversibility, and measurement in quantum mechanics. These studies have been performed in the most controlled manner in quantum optical systems.

As the simplest example, consider the interaction of a two level atom with a single mode of the electromagnetic field



← Optical cavity

Energy in coherently exchanged between the atom and EM mode. This problem was first considered by E.T Jayne + F.W Cummings: Proc IEEE 51, 89 (1963).

Jaynes-Cummings Hamiltonian

The basic Hamiltonian is

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{AF}$$

$$\hat{H}_A = \frac{\hbar\omega_{eg}}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) = \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z$$

$$\hat{H}_F = \hbar\omega_c \hat{a}^\dagger \hat{a}$$

$$\hat{H}_{AF} = -\vec{d}\cdot\vec{E} = -\left(\vec{d}_{eg}\cdot\vec{\epsilon}_c E_{vac}\right)\left(|e\rangle\langle g| + |g\rangle\langle e|\right)\times\left(\hat{a} + \hat{a}^\dagger\right)$$

$$\cong \hbar g\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right) \quad \underline{\text{in R.W.A.}}$$

Where $g = \dfrac{\vec{d}_{eg}\cdot\vec{\epsilon}_c E_{vac}}{\hbar}$ $\qquad E_{vac} = \sqrt{\dfrac{2\pi\hbar\omega_c}{V}}$

$$\Rightarrow \boxed{\hat{H} = \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z + \hbar\omega_c\hat{a}^\dagger\hat{a} + \hbar g\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right)}$$

This simple Hamiltonian exhibits a rich variety of interesting phenomena. Because of its simplicity many things can be solved analytically. With laboratory advances, it became possible to observe them in what can become known a "cavity QED"

Dressed states

When confronted with a new Hamiltonian, the first order of business is to ask about its spectrum: energy eigenstates and eigenvalues.

There is an important conserved quantity

$$\hat{N} = \hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_- = \hat{a}^\dagger \hat{a} + |e\rangle\langle e|$$

$$[\hat{H}, \hat{N}] = [\hat{H}_A + \hat{H}_F + \hat{H}_{AF}, \hat{N}] = [\hat{H}_{AF}, \hat{N}]$$

$$= \hbar g \left[\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+ , \hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_-\right]$$

$$= \hbar g \left(-\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right)$$

$$= 0$$

$\hat{N}$ represents the total excitation of the system (field + atom) which is conserved in the R.W.A. Thus eigenvalues of $\hat{N}$ are "good quantum numbers". They define "manifolds" of states like the manifolds of degenerate sublevels in an atom.

Consider first the trivial case where the coupling constant $g = 0$ (no interaction)

$$\hat{H}_0 = \hat{H}_A + \hat{H}_F = \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z + \hbar\omega_c \hat{a}^\dagger \hat{a}$$

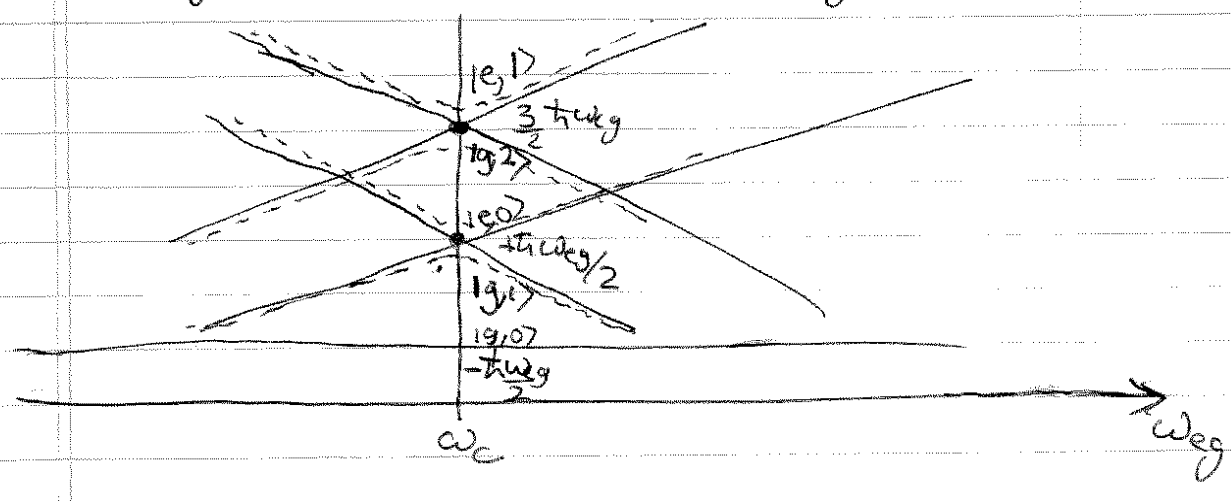
The eigenstates are $|g\rangle|n\rangle$ and $|e\rangle \otimes |n\rangle$

These are known as the "bare states"

The eigenvalues for $|g\rangle \otimes |n\rangle$: $E = -\frac{\hbar\omega_{eg}}{2} + n\hbar\omega_c$

$|e\rangle \otimes |n\rangle$: $E = +\frac{\hbar\omega_{eg}}{2} + n\hbar\omega_c$

Plotting as a function of ω_{eg}



The crossing at ω_c correspond to degenerate states in the same manifold

$N=0$ $|g,0\rangle$

$N=1$ $\{|g,1\rangle, |e,0\rangle\}$

$N=2$ $\{|g,2\rangle, |e,1\rangle\}$

Generally the nth manifold $\{|g,n\rangle, |e,n-1\rangle\}$

Turning on the interaction, states within the same manifold are coupled and those crossings in the energy spectrum become anticrossing as sketched as dotted lines above.

As a matrix, the Hamiltonian in the bare basis is block-diagonal, decomposing into 2×2 blocks for each manifold (except the ground $n=0$)

$$H = \begin{bmatrix} H_0 & & \\ & H_1 & \\ & & H_2 \end{bmatrix}$$

where
$$H_n = \begin{array}{cc} \begin{bmatrix} -\frac{\hbar\omega_{eg}}{2} + n\hbar\omega_c & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & +\frac{\hbar\omega_{eg}}{2} + (n-1)\hbar\omega_c \end{bmatrix} & \begin{array}{l} |g,n\rangle \\ |e,n-1\rangle \end{array} \end{array}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left(n-\tfrac{1}{2}\right)\hbar\omega_c + \begin{bmatrix} \hbar\Delta/2 & \hbar g\sqrt{n} \\ \hbar g\sqrt{n} & -\hbar\Delta/2 \end{bmatrix}$$

$\Delta \equiv \omega_c - \omega_{eg}$ (detuning)

The eigenvalues are

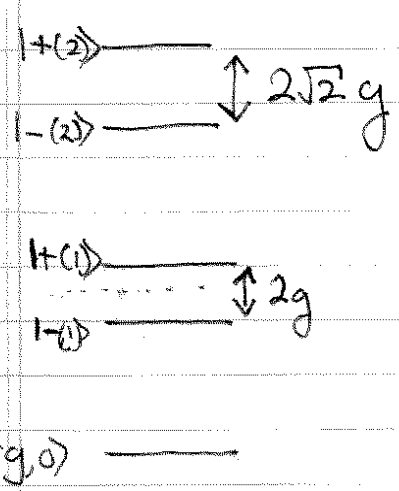
$$\boxed{\begin{array}{l} E_{\pm(n)} = \left(n-\tfrac{1}{2}\right)\hbar\omega_c \pm \frac{\hbar}{2}\sqrt{4g^2 n + \Delta^2} \qquad n = 1, 2 \cdots \\ \text{with eignenstates } |\pm(n)\rangle = \cos\frac{\theta_n}{2}|g,n\rangle + \sin\frac{\theta_n}{2}|e,n+1\rangle \\ \qquad \tan\theta_n = \frac{-\Delta}{2g\sqrt{n}} \end{array}}$$

These states are known as the "dressed-states" in the fully quantum picture. We saw the semi-classical dressed states at the begining of the semester.

On resonance $\omega_{eg} = \omega_c$ $\qquad \Delta = 0$

$$E_{\pm(n)} = \left(n - \tfrac{1}{2}\right)\hbar\omega_{eg} \pm \hbar(2g\sqrt{n})$$

$\Rightarrow$ Jaynes–Cummings ladder



Off resonance the splitting increase

$$\frac{1}{2}\sqrt{4g^2 n + \Delta^2} \equiv \frac{\Omega_n}{2}$$

Whether the upper or lower dressed state in a given manifold is predominantly ground or excited in character depends on detuning

Quantum Rabi flopping and vacuum Rabi splitting

Suppose we start the system in the excited state with no photons. (on resonance)

$$|\psi(0)\rangle = |e,0\rangle$$

This is not a stationary state; it is a superposition of dressed states

$$|e,0\rangle = \frac{|+(1)\rangle - |-(1)\rangle}{\sqrt{2}}$$

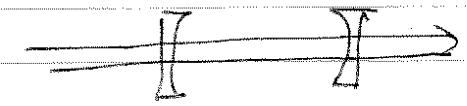
$$\Rightarrow |\psi(t)\rangle = \frac{e^{-E_{+(1)}t/\hbar}|+(1)\rangle - e^{-E_{-(1)}t/\hbar}|-(1)\rangle}{\sqrt{2}}$$

$$\boxed{|\psi(t)\rangle = e^{i(\cdot)}\left(\cos(gt)\,|e,0\rangle - i\sin(gt)\,|g,1\rangle\right)}$$

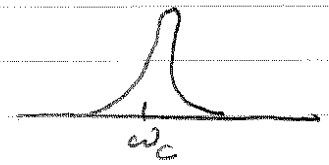
The system oscillates coherently between (atom excited, no photons) and (atom ground, 1 photon)

This is the familar Rabi flopping, but now for the whole system, atom + field.
The single photon Rabi frequency is $2g$.
The splitting between $|+(1)\rangle$ and $|-(1)\rangle$ is sometimes known a "vacuum Rabi splitting"

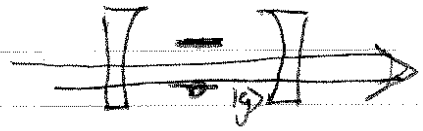
An experiment to observe this is sketched



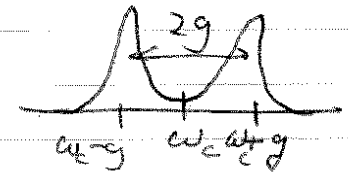
empty cavity



transmission



cavity + ground atom
~ 1 photon



transmission

People sometimes talk of the "atom-cavity molecule"

Transmission resonances are scattering resonances of the quasi bound state

Time evolution operator

Given the analytic solution to the energy eigenvalue problem, we can write an explicit form of the time evolution operator.

Let us specialize to on resonance $\omega_{eg} = \omega_c$.

In the interaction picture $\hat{U}(t) = e^{-i\hat{H}_{int}t/\hbar}$

$$\hat{H}_{int} = \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) \quad \left(\begin{array}{c}\text{still time}\\ \text{independent}\end{array}\right)$$

$$\Rightarrow \quad \hat{U}(t) = \sum_{l=0}^{\infty} \frac{(-ig)^l}{l!} (\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)^l$$

Now since $\hat{\sigma}_+^2 = \hat{\sigma}_-^2 = 0$ we can simplify

If l even $(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)^l = (\hat{a}\hat{a}^\dagger)^{\frac{l}{2}}(\hat{\sigma}_+\hat{\sigma}_-)^{\frac{l}{2}}$
$$+ (\hat{a}^\dagger\hat{a})^{l/2}(\hat{\sigma}_-\hat{\sigma}_+)^{l/2}$$

$$= \left(\sqrt{\hat{a}^\dagger\hat{a}+1}\right)^l |e\rangle\langle e| + \left(\sqrt{\hat{a}^\dagger\hat{a}}\right)^l |g\rangle\langle g|$$

If l odd

$$(\hat{a}\hat{a}^\dagger)^{\frac{l-1}{2}}\hat{a}(\hat{\sigma}_+\hat{\sigma}_-)^{\frac{l-1}{2}}\hat{\sigma}_+ + \hat{a}^\dagger(\hat{a}\hat{a}^\dagger)^{\frac{l-1}{2}}\hat{\sigma}_-(\hat{\sigma}_-\hat{\sigma}_+)^{\frac{l-1}{2}}$$

$$= \frac{\left(\sqrt{\hat{a}^\dagger\hat{a}+1}\right)^l}{\sqrt{\hat{a}^\dagger\hat{a}+1}}\hat{a}|e\rangle\langle g| + \hat{a}^\dagger\frac{\left(\sqrt{\hat{a}^\dagger\hat{a}+1}\right)^l}{\sqrt{\hat{a}^\dagger\hat{a}+1}}|g\rangle\langle e|$$

$$\Rightarrow \boxed{\begin{aligned}\hat{U}(t) = {} & \cos\left(gt\sqrt{\hat{a}^\dagger\hat{a}+1}\right)|e\rangle\langle e| + \cos\left(gt\sqrt{\hat{a}^\dagger\hat{a}}\right)|g\rangle\langle g| \\ & - i\frac{\sin\left(gt\sqrt{\hat{a}^\dagger\hat{a}+1}\right)}{\sqrt{\hat{a}^\dagger\hat{a}+1}}\hat{a}|e\rangle\langle g| - i\hat{a}^\dagger\frac{\sin\left(gt\sqrt{\hat{a}^\dagger\hat{a}+1}\right)}{\sqrt{\hat{a}^\dagger\hat{a}+1}}|g\rangle\langle e|\end{aligned}}$$

Collapse and Revival

To see the difference between the semiclassical and ~~an~~ fully quantum dynamics, consider the closest thing to semiclassical Rabi flopping:

Atom initial in ground state + Field in coherent state

$$|\psi(0)\rangle = |g\rangle \otimes |\alpha\rangle = \sum_n c_n |g\rangle \otimes |n\rangle$$

$\Rightarrow$ In interaction picture $\qquad c_n = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}$

$$|\psi(t)\rangle = \hat{U}(t) \sum_n c_n |g\rangle \otimes |n\rangle$$

$$\boxed{|\psi(t)\rangle = c_0 |g\rangle \otimes |0\rangle + \sum_{n=1}^{\infty} c_n \left(\cos\left(\frac{\Omega_n}{2} t\right) |g\rangle \otimes |n\rangle - i \sin\left(\frac{\Omega_n}{2} t\right) |e\rangle \otimes |n-1\rangle \right) \qquad \Omega_n = 2g\sqrt{n}}$$

The state is a Poissonian distribution of two-state Rabi flopping ~~of~~ between states in each manifold

What is the probability to find the atom in the excited state as a function of time?

$$P_e(t) = \sum_{\substack{n \\ \text{any } n}} P_{e,n}(t) = \sum_n |\langle e,n|\psi(t)\rangle|^2$$

$$= \sum_{n=} \infty |c_n|^2 \sin^2\left(\frac{\Omega_n}{2} t\right)$$

$$\boxed{P_e(t) = \sum_{n=1}^{\infty} \frac{\bar{n}^n e^{-\bar{n}}}{n!} \sin^2(g\sqrt{n}\, t)}$$

This is similar to the problem of inhomogeneous broadening we saw at the begining of the semester.

Because we are superposing a collection of sinusoids with different frequencies the oscillation will decay. The decay (or collapse) time is determined by the spread in frequencies.

Given a spread in photon number $\Delta n = \sqrt{\bar{n}}$, the spread in frequencies is

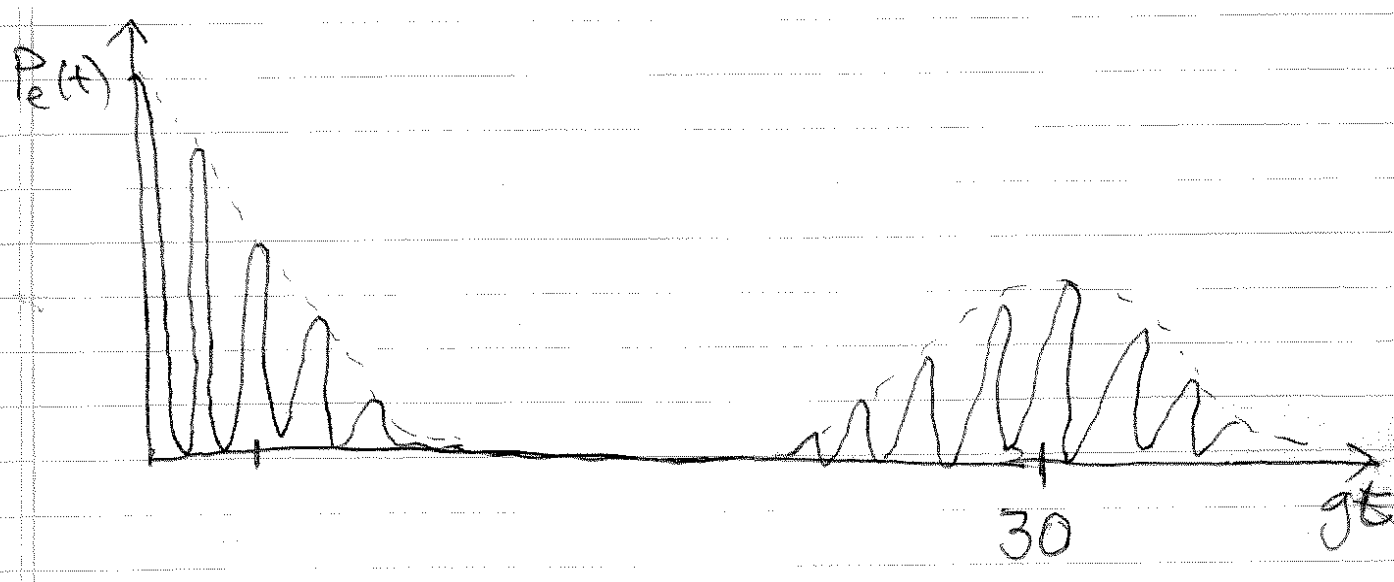
$$\Delta\Omega \sim |\Omega_{n+\sqrt{n}} - \Omega_{n-\sqrt{n}}|$$

$$\Rightarrow \text{Collapse time } t_c \sim \frac{1}{\Delta\Omega} \sim \frac{1}{2g\sqrt{n+\sqrt{n}} - 2g\sqrt{n-\sqrt{n}}}$$

For $\bar{n} \gg \sqrt{\bar{n}}$ $\quad t_c \sim \frac{1}{2g\sqrt{n}} \frac{1}{\left(1+\frac{1}{2}\frac{\sqrt{n}}{n}\right) - \left(1-\frac{\sqrt{n}}{2n}\right)}$

$$\Rightarrow \boxed{t_c \sim \frac{1}{2g}}$$

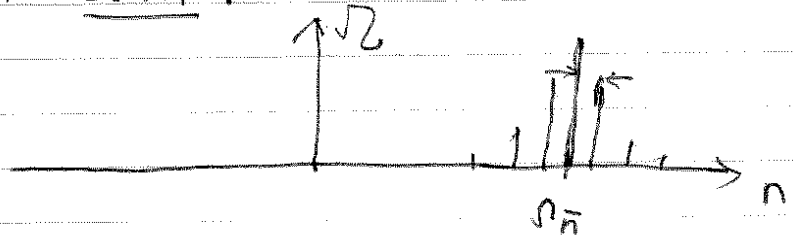
A plot of the solution for $\bar{n} = 25$



The solution shows the expected decay due to the spread of oscillation frequencies. In addition there is a <u>revival</u> in the Rabi oscillation.

This is a completely nonclassical effect due to the discrete nature of the photon. Classically intensity fluctuations would lead to a decay of Rabi oscillation due to a fluctuation in $\Omega = \vec{d}\cdot\vec{E}/\hbar$.

If these are quantum fluctuations, then there are only discrete possibilities for the amplitude. Instead of a Fourier <u>integral</u> over sinusoids, we have a Fourier <u>sum</u>.



At times such that $(\Omega_{\bar{n}} - \Omega_{\bar{n}-1})\, t_r \cong 2\pi m$

we expect a rephasing and "revival"

$$\Rightarrow t_r \cong \frac{2\pi m}{2g\sqrt{n} - 2g\sqrt{n-1}} \cong \frac{\pi m}{g\sqrt{n}} \frac{1}{\left(1-\left(1-\frac{1}{2n}\right)\right)}$$

$$\Rightarrow \boxed{t_r \cong \frac{2\pi m\sqrt{n}}{g}}$$

Recovering the classical limit

We expect classical behavior as $\bar{n} \Rightarrow \infty$.

The mean Rabi frequency

$$\Omega_{\bar{n}} = 2g\bar{n} = 2\bar{n}\sqrt{\frac{2\pi\hbar\omega}{V}}\, d_{eg}/\hbar$$

$$= \frac{\left(\sqrt{\frac{8\pi\hbar\omega}{V}}\,\bar{n}\right) d_{eg}}{\hbar} = \frac{d_{eg}E}{\hbar}$$

as $\bar{n} \to \infty$ $\Omega_{\bar{n}} \gg g$ $\Rightarrow$ Many oscillations before collapse. In fact, in the classical limit $V \to \infty$, spontaneous emission will always lead to decay of Rabi oscillation before the collapse due to fluctuations in the mode