Lecture 17: The Jaynes-Cummings Problem

Besides the spontaneous emission problem we have considered two kinds of semiclassical problems:

- Quantum atom interacting with classical field (e.g., Rabi flopping)
- Quantum field interacting with classical matter (e.g., Parametric downconversion)

As we know the world to be quantum mechanical at its foundation, we would like to see these theories as limits to a more fundamental model. We thus begin our exploration of coupled quantum systems. This investigation will bring to the very important problem of open quantum systems and its relation to dissipation, decoherence, irreversibility, and measurement in quantum mechanics. These studies have been performed in the most controlled manner in quantum optical systems.

As the simplest example, consider the interaction of a two-level atom with a single mode of the electromagnetic field

\[
\begin{array}{c}
\hline
\hline
\text{\[\begin{array}{c}
\alpha, \beta
\end{array}\]}
\hline
\hline
\end{array}
\]

\text{Optical cavity}

Energy in coherently exchanged between the atom and EM mode. This problem was first considered by E.T. Jaynes & F.W. Cummings; Proc. IEEE 51, 89 (1963).
Jaynes-Cummings Hamiltonian

The basic Hamiltonian is

\[ \hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{AF} \]

\[ \hat{H}_A = \frac{\hbar \omega_{eg}}{2} (|e\rangle\langle e| - |g\rangle\langle g|) = \frac{\hbar \omega_g}{2} \hat{\sigma}_z \]

\[ \hat{H}_F = \hbar \omega_c \hat{a}^+ \hat{a} \]

\[ \hat{H}_{AF} = -\hat{\mathbf{J}} \cdot \mathbf{E} = - (\hat{\mathbf{J}} \cdot \hat{\mathbf{E}}_{\text{vac}}) g (\mathbf{e} \times \mathbf{g}) \cdot (|e\rangle \langle g| + |g\rangle \langle e|) \]

\[ \approx \hbar g \left( \hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^+ \right) \quad \text{in R.W.A.} \]

Where \( g = \frac{\text{deg} \cdot \hat{\mathbf{E}}_{\text{vac}}}{\hbar} \quad \text{Evac} = \sqrt{\frac{2 \pi \hbar \omega_c}{V}} \)

\[ \Rightarrow \hat{H} = \frac{\hbar \omega_g}{2} \hat{\sigma}_z + \hbar \omega_c \hat{a}^+ \hat{a} + \hbar g \left( \hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^+ \right) \]

This simple Hamiltonian exhibits a rich variety of interesting phenomena. Because of its simplicity, many things can be solved analytically. With laboratory advances, it became possible to observe them in what can become known a "cavity QED."
Dressed states

When confronted with a new Hamiltonian, the first order of business is to ask about its spectrum: energy eigenstates and eigenvalues.

There is an important conserved quantity

\[ \hat{N} = \hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_- = \hat{a}^\dagger \hat{a} + |e\rangle \langle e| \]

\[ [\hat{A}, \hat{N}] = [\hat{A}_A + \hat{A}_F + \hat{H}_{AF}, \hat{N}] = [\hat{H}_{AF}, \hat{N}] \]

\[ = \hbar g \left[ \hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+ , \hat{a}^\dagger \hat{a} + \hat{\sigma}_- \hat{\sigma}_+ \right] \]

\[ = \hbar g \left( -\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+ + \hat{\sigma}_- \hat{\sigma}_+ + \hat{\sigma}_- \hat{\sigma}_+ \right) \]

\[ = 0 \]

\[ \hat{N} \] represents the total excitation of the system (field+atom) which is conserved in the R.W.A. Thus eigenvalues of \( \hat{N} \) are "good quantum numbers". They define "manifolds" of states like the manifolds of magnetic sublevels in an atom.

Consider first the trivial case where the coupling constant \( g = 0 \) (no interaction)

\[ \hat{H}_0 = \hat{A}_A + \hat{A}_F = \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \hbar \omega_c \hat{a}^\dagger \hat{a} \]

The eigenstates are \( |g\rangle |n\rangle \) and \( |e\rangle |n\rangle \)

These are known as the "bare states"
The eigenvalues for $|g\rangle @ |n\rangle$: 
$E = -\frac{\hbar \omega}{2} + n\hbar \omega_c$

$|e\rangle @ |n\rangle$: 
$E = \frac{\hbar \omega}{2} + n\hbar \omega_c$

Plotting as a function of $\omega_c$

The crossing at $\omega_c$ correspond to degenerate states in the same manifold.

$N=0$ $|g, 0\rangle$

$N=1$ $\Sigma |g, 1\rangle + |e, 0\rangle$

$N=2$ $\Sigma |g, 2\rangle + |e, 1\rangle$

Generally the $n$th manifold $\Sigma |g, n\rangle + |e, n-1\rangle$

Turning on the interaction, states within the same manifold are coupled and those crossings in the energy spectrum become anticrossing as sketched as dotted lines above.
As a matrix, the Hamiltonian in the bare basis is block-diagonal, decomposing into $2 \times 2$ blocks for each manifold (except the ground $n=0$)

$$H = \begin{bmatrix} H_0 & \frac{t_g \sqrt{r}}{\Delta} \\ \frac{t_g \sqrt{r}}{\Delta} & H_2 \end{bmatrix}$$

where

$$H_n = \begin{bmatrix} -\frac{\hbar \omega_y + n \hbar \omega_c}{2} & t_g \sqrt{r} \\ \frac{t_g \sqrt{r}}{\Delta_n} & \frac{\hbar \omega_y + (n-1) \hbar \omega_c}{2} \end{bmatrix} \left| g, n \right> \left< g, n - 1 \right|$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left( -\frac{\hbar \omega_c}{2} \right)^n + \left[ \begin{bmatrix} \Delta_n^2 & t_g \sqrt{r} \\ t_g \sqrt{r} & -\Delta_n^2 \end{bmatrix} \right] \Delta = \omega_c - \omega_g$$

The eigenvalues are

$$E_{\pm n} = \left( n - \frac{1}{2} \right) \frac{\hbar \omega_c}{2} \pm \frac{1}{2} \sqrt{4 g^2 n + \Delta^2} \quad n = 1, 2, \ldots$$

with eigenstates $\left| \pm (n) \right> = \cos \frac{\Theta_n}{2} \left| g, n \right> + \sin \frac{\Theta_n}{2} \left| e, n+1 \right>$

$$\tan \Theta_n = -\frac{\Delta}{2 g \sqrt{n}}$$

These states are known as the "dressed states" in the fully quantum picture. We saw the semi-classical dressed states at the beginning of the semester.
At resonance \( \omega_g = \omega_0 \) \( \Delta = 0 \)

\[ E_{\pm n} = (n - \frac{1}{2}) \hbar \omega_g \pm \hbar (2g \sqrt{n}) \]

Jaynes-Cummings ladder:

\[ +: \quad 1 \quad \uparrow 2 \sqrt{2} g \]

\[ 1 \quad \uparrow 2g \]

\[ 1 \quad \uparrow 2g \]

\[ 0 \]

Off-resonance

\[ \frac{1}{2} \sqrt{4g^2 n + \Delta^2} = \frac{\hbar \omega_0}{2} \]

Whether the upper or lower dressed state in a given manifold is predominant ground or excited in character depends on detuning.

Quantum Rabi damping and vacuum Rabi splitting

Suppose we start the system in the excited state with no photons.

This is not a stationary state; it is a superposition of dressed states.

\[ |0,0 \rangle = \frac{|1+0 \rangle - |1-0 \rangle}{\sqrt{2}} \]

\[ |2+0 \rangle = e^{-\frac{E_{+0} t}{\hbar}} |1+0 \rangle - e^{-\frac{E_{-0} t}{\hbar}} |1-0 \rangle \]

\[ \Rightarrow |2+0 \rangle = e^{i\omega_0 t} |0,0 \rangle - i \sin(\omega_0 t) |1g,1 \rangle \]
The system oscillates coherently between (atom, no photon) and (atom, ground, 1 photon).

This is the familiar Rabi flopping, but now for the whole system, atom + field. The single photon Rabi frequency is $2g$.

The splitting between $|+(1\rangle$ and $|-(1\rangle$ is sometimes known as "vacuum Rabi splitting."

An experiment to observe this is sketched:

- [Diagram of empty cavity and transmission]
- [Diagram of cavity + ground atom and transmission]

People sometimes talk of the "atom-cavity molecule."

Transmission resonances are scattering resonances of the quasi bound state.
Time evolution operator

Given the analytic solution to the energy eigenvalue problem, we can write an explicit form of the time evolution operator.

Let us specialize to an resonance \( \omega_g = \omega_c \).

In the interaction picture \( \hat{U}(t) = e^{-\frac{i \hat{H}_{\text{int}} t}{\hbar}} \)

\[ \hat{H}_{\text{int}} = \hbar g (\hat{a} \hat{\sigma}_+ + \hat{a}^+ \hat{\sigma}_-) \]  

(\text{small time independent})

\[ \Rightarrow \hat{U}(t) = \sum_{l=0}^{\infty} \frac{(i g)^l}{l!} (\hat{a} \hat{\sigma}_+ + \hat{a}^+ \hat{\sigma}_-) \]

Now since \( \hat{\sigma}_+^2 = \hat{\sigma}_- = 0 \) we can simplify.

- If \( l \) even

\[ (\hat{a} \hat{\sigma}_+ + \hat{a}^+ \hat{\sigma}_-) = (\hat{a} \hat{\sigma}_+)^{\frac{l}{2}} (\hat{\sigma}_+ \hat{\sigma}_-) \]

\[ = (\sqrt{\hat{a}^+ \hat{a} + 1})^l e^{i g l} + (\hat{a}^+ \hat{a})^{\frac{l}{2}} 1 g > g \]

- If \( l \) odd

\[ (\hat{a} \hat{\sigma}_+)^{\frac{l-1}{2}} \hat{a} (\hat{\sigma}_+ \hat{\sigma}_-) \hat{a}^+ (\hat{a} \hat{\sigma}_+)^{\frac{l}{2}} \hat{\sigma}_- (\hat{\sigma}_+ \hat{\sigma}_-) \]

\[ = \frac{(\sqrt{\hat{a}^+ \hat{a} + 1})^l}{\sqrt{\hat{a}^+ \hat{a} + 1}} \hat{a}^{l-1} e^{i g l} + \hat{a}^+ \frac{(\sqrt{\hat{a}^+ \hat{a} + 1})^{l+1}}{\sqrt{\hat{a}^+ \hat{a} + 1}} 1 g > c \]

\[ \Rightarrow \hat{U}(t) = \cos \left( g t \sqrt{\hat{a}^+ \hat{a} + 1} \right) e^{i g t} + \cos \left( g t \sqrt{\hat{a}^+ \hat{a}} \right) 1 g > g \]

\[ -i \frac{\sin \left( g t \sqrt{\hat{a}^+ \hat{a} + 1} \right)}{\sqrt{\hat{a}^+ \hat{a} + 1}} \hat{a} 1 g < c - i \frac{\sqrt{\hat{a}^+ \hat{a}}}{\sqrt{\hat{a}^+ \hat{a} + 1}} 1 g < g \]
Collapse and Revival

To see the difference between the semiclassical and a full quantum dynamics, consider the closest theory to semiclassical Rabi flopping:

Atom initial in ground state $\rightarrow$ Field in coherent state

$$|\psi(0)> = |g> \otimes |0> = \sum_n c_n |g> \otimes |n>$$

$$\Rightarrow$$ In interaction picture

$$|\psi(t) > = \hat{U}(t) \sum_n c_n |g> \otimes |n>$$

$$|\psi(t) = c_0 |g \otimes |0> + \sum_{n=1}^{\infty} c_n \left( \cos \left( \frac{\Omega n}{2} t \right) |g \otimes |n> - i \sin \left( \frac{\Omega n}{2} t \right) |e \otimes |n> \right)$$

$$\Omega_n = 2g \sqrt{n}$$

The state is a Poissonian distribution of two-state Rabi floppings between states in each manifold.

What is the probability to find the atom in the excited state as a function of time?

$$P_e(t) = \sum_{n \in \mathbb{R}} P_{e,n}(t) = \sum_{n} K_{e,n} |\psi(t)> |^2$$

any $n$

$$= \sum_{n=1}^{\infty} |c_n|^2 \sin^2 \left( \frac{\Omega_n}{2} t \right)$$

$$P_e(t) = \sum_{n=1}^{\infty} \frac{n! e^{-n}}{n!} \sin^2 \left( \frac{g \sqrt{n} t}{2} \right)$$
This is similar to the problem of inhomogeneous broadening we saw at the beginning of the semester.

Because we are superposing a collection of sinusoids with different frequencies, the oscillation will decay. The decay (or collapse) time is determined by the spread in frequencies.

Given a spread in photon number \( \Delta n = \sqrt{n} \), the spread in frequencies is

\[
\Delta \Omega \sim \left| \Omega_{n+\sqrt{n}} - \Omega_{n-\sqrt{n}} \right|
\]

\[ \Rightarrow \text{collapse time } t_c \sim \frac{1}{\Delta \Omega} \sim \frac{1}{2g \sqrt{n+\sqrt{n} - 2g \sqrt{n-\sqrt{n}}}} \]

For \( n \gg \sqrt{n} \)

\[ t_c \sim \frac{1}{2g \sqrt{n} \left(1+\frac{1}{2n}\right) - \left(1-\frac{1}{2n}\right)} \]

\[ \Rightarrow t_c \sim \frac{1}{2g \sqrt{n}} \]

A plot of the solution for \( n = 25 \)

\[ P_e(t) \]
The solution shows the expected decay due to the spread of oscillation frequencies. In addition, there is a revival in the Rabi oscillation. This is a completely nonclassical effect due to the discrete nature of the photon. Classically, intensity fluctuations would lead to a decay of Rabi oscillation due to a fluctuation in $\Delta = \frac{1}{\hbar} E_n \Delta$.

If there are quantum fluctuations, then there are only discrete possibilities for the amplitude. Instead of a Fourier integral over sinusoids, we have a Fourier sum.

At times such that $(\Omega_n - \Omega_{n-1}) t \approx 2\pi m$

we expect a rephasing and "revival"

\[ \Rightarrow \quad t_r \approx \frac{2\pi m}{2gJ_n - 2gJ_{n-1}} \]

\[ \approx \frac{\pi m}{g\sqrt{n}} \frac{1}{(1 - (1 - \frac{1}{2n}))} \]

\[ \Rightarrow \quad t_r \approx \frac{2\pi m \sqrt{n}}{g} \]
Recovering the classical limit

We expect classical behavior as \( \bar{n} \to \infty \).

The mean Rabi frequency

\[
\sqrt{2\bar{n}} = 2g \bar{n} = 2\bar{n} \sqrt{\frac{2\pi k \omega}{V}} \, \text{deg} \, \frac{1}{\tau}
\]

\[
= \left( \frac{8\pi \hbar \omega}{V} \bar{n} \right) \, \text{deg} \, \frac{1}{\tau} = \frac{\text{deg} \, E}{\tau}
\]

as \( \bar{n} \to \infty \), \( \sqrt{2\bar{n}} \gg g \) \( \Rightarrow \) many oscillations before collapse. In fact, in the classical limit \( V \to \infty \), spontaneous emission will always lead to decay of Rabi oscillation before the collapse due to fluctuations in the mode.