

Lecture 18: Introduction to irreversibility: System-Reservoir interaction

We have seen two very different kinds of dynamics in the evolution of an atom coupled to the quantized electromagnetic field

- Spontaneous emission: Atom in free space, initially prepared in excited state with field in the vacuum

$$|\psi(0)\rangle = |e\rangle \otimes |vac\rangle$$

At later time the probability to find excited atom

$$P_e(t) = e^{-\Gamma t}$$

$$\Gamma = \frac{4}{3} \text{deg}^2 \frac{k^3}{\hbar} = \frac{1}{\tau_{1/2}}$$

- Jaynes-Cummings: Atom in a closed cavity initially prepared in excited state with field in vacuum

$$|\psi(0)\rangle = |e\rangle \otimes |vac\rangle$$

At later time

$$P_e(t) = \cos^2(gt)$$

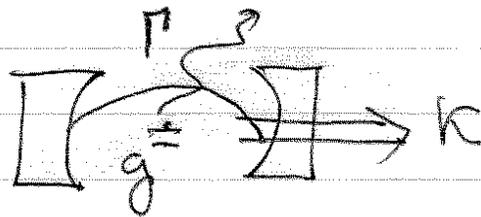
$$g = \frac{1}{\hbar} \sqrt{\frac{2\pi\hbar\omega}{V}} \text{deg}$$

These represent two extremes:

- Reversible behavior (JC problem)
- Irreversible behavior (spontaneous emission)

Time reversibility $t \rightarrow -t$

In a certain sense the irreversible ~~is~~ behavior is more generic; it is characteristic of what is typically observed in the laboratory. For example, ~~the~~ the J-C model is a idealization. Rabi flopping between $|e\rangle \otimes |0\rangle$ and $|g\rangle \otimes |1\rangle$ does not go on indefinitely. Eventually, the photon will be spontaneously emitted out the sides, or absorbed/transmitted through mirror



The J-C model only hold approximately when $g \gg \kappa, \Gamma$

This is usually called the "strong coupling" regime.

Whereas the ideal J-C model is a closed quantum system, the real world is not.

We thus begin our exploration of open quantum systems. This has important implications for the description of irreversibility, dissipation, decoherence, and quantum measurement. Though by no means unique to quantum optical systems, it is in quantum optics where some of the clearest and most controlled studies have been performed.

Unitary vs. nonunitary

The appearance of irreversible behavior is somewhat paradoxical given that the Schrödinger equation is time reversal invariant. That is Hamiltonian dynamics generates unitary transformations

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle$$

$$\Rightarrow |\psi(t_0)\rangle = \hat{U}^{-1}(t, t_0) |\psi(t)\rangle = \hat{U}^\dagger(t, t_0) |\psi(t)\rangle \\ = \hat{U}(t_0; t) |\psi(t)\rangle$$

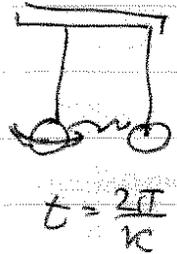
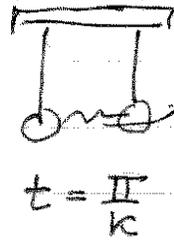
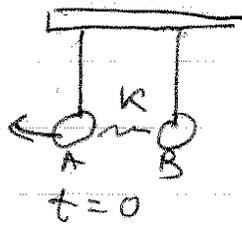
Given that the fundamental theory generates time reversible unitary dynamics, from where does irreversible behavior arise?

This paradox is not, of course, specific to quantum theory. In classical physics the microscopic laws are Hamiltonian, time reversible. Observations show irreversible behavior, consistent with the laws of thermodynamics. These are reconciled through statistical physics. There is thus an intimate relationship between our problem and nonequilibrium statistical mechanics.

Irreversibility arising an effective theory in a "macroscopic limit".

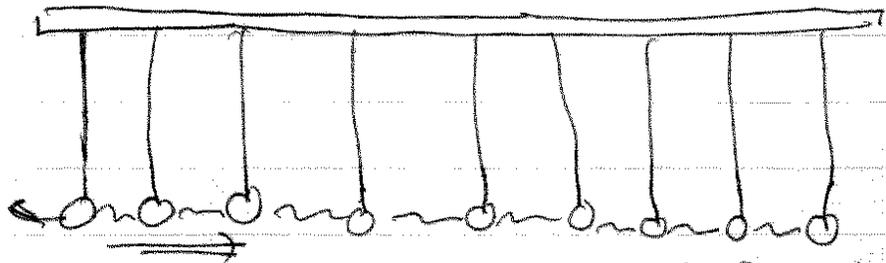
Simple picture of reversible vs. irreversible

2 Coupled pendula:



Energy exchanged periodically between A & B

Many coupled pendula



energy propagated for long time

Elastic rod (infinite in length)

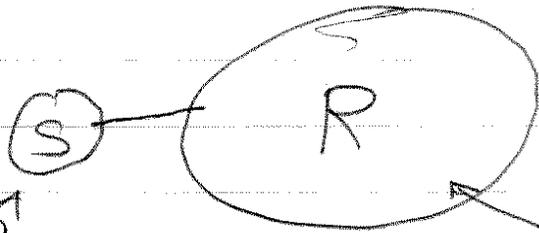


\Rightarrow Infinite # of normal modes
(uncountably)

\Rightarrow Recurrence time $\rightarrow \infty$

\Rightarrow Effectively irreversible

System-Reservoir picture



- "small system"
 - few degrees of freedom (d.o.f)

- "larger reservoir"
 - many (approaching continuum) of d.o.f.

In this picture we view the system of interest in contact with a "bath" or "reservoir" in some equilibrium state. The massive bath is like the uncountably infinite collection of pendula; - once an excitation flows in, it doesn't flow back to the system.

We are ultimately interested in the quantum description, the system-reservoir interaction not only has the effect of dissipation of energy (as in classical physics), but of damping quantum coherence. This process, known as "decoherence" removes certain quantum interference phenomena very quickly, the removal of coherence changes quantum superpositions into statistical mixtures, i.e. pure state vectors to mixed density matrices

Bipartite system

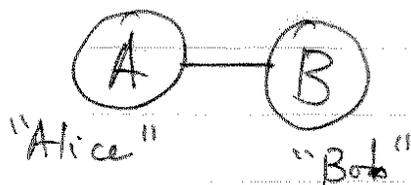
The system-reservoir description is "bipartite"
(two parts)

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR} \leftarrow \text{interaction}$$

↙ ↘

System Hamiltonian Reservoir Hamiltonian

We are interested in the evolution of the system when we ignore, or are unable to keep track of, the state of the reservoir. Let us consider such system generically.



⌘ Suppose $\Psi(\vec{x}_A, \vec{x}_B)$ is the joint wavefunction. What is the probability of find Alice at \vec{x}_A irrespective of Bob?

Joint probability ^{density} $P_{AB}(\vec{x}_A, \vec{x}_B) = |\Psi(\vec{x}_A, \vec{x}_B)|^2$

Marginal prob. density $P_A(\vec{x}_A) = \int d^3\vec{x}_B |\Psi(\vec{x}_A, \vec{x}_B)|^2$

Note in general $P_A(\vec{x}_A) \neq |\phi_A(\vec{x}_A)|^2$ for any $\phi_A(\vec{x}_A)$
Alice's state is not generally describable by a wave function.

Exception, if $|\Psi(\vec{x}_A, \vec{x}_B)\rangle = \phi_A(\vec{x}_A) \chi_A(\vec{x}_B)$
(separable)

$$\Rightarrow P_{AB}(\vec{x}_A, \vec{x}_B) = P_A(\vec{x}_A) P_B(\vec{x}_B)$$

$$|\Psi(\vec{x}_A, \vec{x}_B)|^2 = |\phi_A(\vec{x}_A)|^2 |\phi_B(\vec{x}_B)|^2$$

Abstract picture:

Hilbert space for joint A+B system

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

↑
"Kronecker tensor product"

If $\{|e_i\rangle; i=1, 2, \dots, d_A\}$ is a basis for \mathcal{H}_A
(dimension d_A)

$\{|d_j\rangle; j=1, 2, \dots, d_B\}$ is a basis for \mathcal{H}_B
(dimension d_B)

$\{|u_{ij}\rangle = |e_i\rangle \otimes |d_j\rangle\}$ is a basis for \mathcal{H}_{AB}
(dimension $d_A d_B$)

$$|\Psi_{AB}\rangle = \sum_{ij} c_{ij} |u_{ij}\rangle$$

$$c_{ij} = \langle u_{ij} | \Psi_{AB} \rangle$$

$$= (\langle e_i | \otimes \langle d_j |) |\Psi_{AB}\rangle$$

In matrix notation (e.g. $d_1 = d_2 = 2$)

$$|\psi_A\rangle = \begin{bmatrix} c_1^A \\ c_2^A \end{bmatrix}$$

$$|\psi_B\rangle = \begin{bmatrix} d_1^B \\ d_2^B \end{bmatrix}$$

$$|\psi_A\rangle \otimes |\psi_B\rangle = \begin{bmatrix} c_1^A d_1^B \\ c_1^A d_2^B \\ c_2^A d_1^B \\ c_2^A d_2^B \end{bmatrix} \quad (4 \text{ dim vector})$$

Operators on \mathcal{H}_{AB}

Suppose \hat{A} is an operator on \mathcal{H}_A
 \hat{B} is an operator on \mathcal{H}_B

We can define $\hat{O} = \hat{A} \otimes \hat{B}$

$$\begin{aligned}\hat{O}|\Psi_{AB}\rangle &= \hat{A} \otimes \hat{B} \left(\sum_{ij} c_{ij} |e_i\rangle \otimes |d_j\rangle \right) \\ &= \sum_{ij} c_{ij} (\hat{A}|e_i\rangle) \otimes (\hat{B}|d_j\rangle)\end{aligned}$$

→ As a matrix $\langle i'j' | \hat{O} | ij \rangle = \langle i' | \hat{A} | i \rangle$

e.g. $d_1 = d_2 = 2$ $\hat{A} = \hat{\sigma}_x$ $\hat{B} = \hat{\sigma}_z$

In standard basis $|\pm_z\rangle_A \otimes |\pm_z\rangle_B$

$$\hat{\sigma}_x \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{\sigma}_z \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \hat{\sigma}_x \otimes \hat{\sigma}_z \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Note if operator only acts on \mathcal{H}_A then on joint space $\hat{A} \Rightarrow \hat{A} \otimes \hat{I}_B \leftarrow \text{identity}$

Expectation values and the partial trace

Let $|\Psi_{AB}\rangle = \sum_{ij} c_{ij} |e_i\rangle \otimes |d_j\rangle$ be a state on \mathcal{H}_{AB}

\hat{O} some observable on \mathcal{H}_{AB}

$$\Rightarrow \langle \hat{O} \rangle = \langle \Psi_{AB} | \hat{O} | \Psi_{AB} \rangle$$

Suppose $\hat{O} = \hat{A} \otimes \hat{I}_B$ (observable acting on \hat{A} only)

$$\begin{aligned} \Rightarrow \langle \hat{O} \rangle &= \langle \Psi_{AB} | \hat{A} \otimes \hat{I}_B | \Psi_{AB} \rangle \\ &= \sum_{i,j,i',j'} c_{ij}^* c_{i'j'} \langle e_i | \hat{A} | e_{i'} \rangle \underbrace{\langle d_j | \hat{I}_B | d_{j'} \rangle}_{\delta_{jj'}} \\ &= \sum_{i,i'} \underbrace{\left(\sum_j c_{ij}^* c_{i'j} \right)}_{\hat{\rho}^A} \langle e_i | \hat{A} | e_{i'} \rangle \end{aligned}$$

$\equiv \hat{\rho}^A$ For reasons to become clear

$$\Rightarrow \langle \hat{O} \rangle = \text{Tr}(\hat{\rho}^A \hat{A}) = \langle \hat{A} \rangle$$

$$\text{where } \hat{\rho}^A = \sum_j c_{ij}^* c_{i'j} |e_i\rangle \langle e_{i'}|$$

$\hat{\rho}^A$ describes the state of Alice, ignoring Bob. It is the "reduced" or "marginal" density operator.

Generally, suppose $\hat{\rho}_{AB}$ is the state of \mathcal{H}_{AB}

$$\text{e.g. } \hat{\rho}_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}| = \sum_{ij} c_{ij}^* c_{ij} |ij\rangle\langle ij| \\ (|ij\rangle \equiv |e_i\rangle \otimes |d_j\rangle)$$

$$\Rightarrow \langle \hat{O} \rangle = \text{Tr}_{AB}(\hat{O} \hat{\rho}_{AB}) \text{ by basic axioms of Q.M.} \\ = \sum_{ij} \langle ij | \hat{O} \hat{\rho}_{AB} | ij \rangle$$

$$\text{If } \hat{O} = \hat{A} \otimes \hat{I}_B$$

$$\Rightarrow \langle \hat{O} \rangle = \sum_{ij} \langle e_i | \hat{A} \langle d_j | \hat{\rho}_{AB} | d_j \rangle | e_i \rangle \\ = \sum_i \langle e_i | \hat{A} \underbrace{\sum_j \langle d_j | \hat{\rho}_{AB} | d_j \rangle}_{\equiv \hat{\rho}_A} | e_i \rangle$$

$$\Rightarrow \hat{\rho}_A = \text{Tr}_B(\hat{\rho}_{AB}) = \sum_j \langle d_j | \hat{\rho}_{AB} | d_j \rangle$$

Partial trace \Rightarrow marginal

$$\Rightarrow \langle \hat{A} \rangle = \text{Tr}_A(\hat{A} \hat{\rho}_A)$$

Partial trace and increase in entropy

Consider again $d_1 = d_2 = 2$ (two spin $1/2$ particles)

$$\text{Let } |\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|+\rangle_A \otimes |-\rangle_B - |-\rangle_A \otimes |+\rangle_B) \quad \text{"Spin Singlet"}$$

What is the state of Alice's spin alone?

$$\hat{\rho}_{AB} = \frac{1}{2} (|+\rangle_A \langle +| \otimes |-\rangle_B \langle -| + |-\rangle_A \langle -| \otimes |+\rangle_B \langle +| \\ - |+\rangle_A \langle -| \otimes |-\rangle_B \langle +| - |-\rangle_A \langle +| \otimes |+\rangle_B \langle -|)$$

Marginal $\hat{\rho}_A = \text{Tr}_B(\hat{\rho}_{AB}) = \sum_{m_B = \pm} \langle m_B | \hat{\rho}_{AB} | m_B \rangle$

$$= \frac{1}{2} (|+\rangle_A \langle +| + |-\rangle_A \langle -|)$$

$$\stackrel{\circ}{=} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \hat{\rho}_A$ is a completely mixed state

i.e. 50-50 statistical mixture of $| \pm \rangle$

In contrast, the joint system was pure

\Rightarrow Taking partial trace can reduce purity (increase entropy) of the marginals.

Information contained in correlations between A and B

Entanglement

Given pure state $|\Psi_{AB}\rangle$ when is $\hat{\rho}_A$ pure?

Answer if and only if $|\Psi_{AB}\rangle = |\phi_A\rangle \otimes |\chi_B\rangle$
for some $|\phi_A\rangle$ and $|\chi_B\rangle$

$$\text{If } |\Psi_{AB}\rangle = |\phi_A\rangle \otimes |\chi_B\rangle \Rightarrow \hat{\rho}_{AB} = |\phi_A\rangle\langle\phi_A| \otimes |\chi_B\rangle\langle\chi_B|$$

$$\begin{aligned} \Rightarrow \text{Tr}_B(\hat{\rho}_{AB}) &= |\phi_A\rangle\langle\phi_A| \underbrace{\text{Tr}_B(|\chi_B\rangle\langle\chi_B|)}_{=1} \\ &= |\phi_A\rangle\langle\phi_A| \end{aligned}$$

If $|\Psi_{AB}\rangle \neq |\phi_A\rangle \otimes |\chi_B\rangle$ it is said to
be an entangled state. \neq

$\Rightarrow \hat{\rho}_A$ and $\hat{\rho}_B$ are somewhat mixed

Maximal entanglement \Rightarrow maximal mixing of
marginals

Recall, Von Neuman entropy

$$S = -\text{Tr}(\hat{\rho} \log \hat{\rho}) = \sum_{\lambda} -\lambda \ln \lambda$$

where $\{\lambda\}$ eigenvalues of $\hat{\rho}$

Pure state $\lambda=1$ and 0 for all states

$$\Rightarrow S = 0 \quad \text{pure state}$$

Max mixed $\lambda = \frac{1}{d} \forall$ eigenvectors $d = \dim$

$$\Rightarrow S = \log d \quad \text{max mixed}$$

Schmidt decomposition

For the case of a pure bipartite state we can summarize everything related to entanglement through a special representation known as the Schmidt decomposition.

Let $\{|a\rangle\}$ be an ^{orthonormal} basis for \mathcal{H}_A , dimension d_A
 $\{|b\rangle\}$ be an orthonormal bases for \mathcal{H}_B , dim d_B

Given an arbitrary state $|\Psi_{AB}\rangle$ it can be expanded in the product basis

$$|\Psi_{AB}\rangle = \sum_{a=1}^{d_A} \sum_{b=1}^{d_B} C_{ab} |a\rangle \otimes |b\rangle$$

The question is, how is the entanglement of the state related to the expansion coefficient C_{ab} ?

To solve this, we appeal to the singular-value decomposition of matrix theory. The numbers C_{ab} are elements of a $d_A \times d_B$ matrix. Any matrix C can be decomposed ~~as~~ as

$$C = U^T D V$$

where U and V are unitary matrices of dimension $d_A \times d_A$ and $d_B \times d_B$

And D is a diagonal matrix $D_{ij} = \lambda_i \delta_{ij}$
where the # of non zero λ 's $\leq \min(d_A, d_B)$

The numbers $\{\lambda_i\}$ are known as the singular values of the matrix. Let us plug in the SVD into the expansion of $|\Psi_{AB}\rangle$: $c_{ab} = (u^T)_{ai} \lambda_i v_{ib}$

$$|\Psi_{AB}\rangle = \sum_{i=1}^{\#\lambda's} \lambda_i \left(\sum_{a=1}^{d_A} u_{ia} |a\rangle \right) \otimes \left(\sum_{b=1}^{d_B} v_{ib} |b\rangle \right)$$

$$\equiv |u_i\rangle_A \otimes |v_i\rangle_B$$

The sets $\{|u_i\rangle_A\}$ $\{|v_i\rangle_B\}$ are new orthonormal bases for \mathcal{H}_A and \mathcal{H}_B respectively

The special expansion

$$|\Psi_{AB}\rangle = \sum_{i=1}^{\#\lambda's} \lambda_i |u_i\rangle_A \otimes |v_i\rangle_B$$

is known as the Schmidt decomposition.

It differs from the arbitrary expansion on the previous page in that it is summed over one index.

The bases $\{|u_i\rangle_A\}$ $\{|v_i\rangle_B\}$ are thus special. They are known as the Schmidt basis, ~~and~~ ^{and} they depend on the state $|\Psi_{AB}\rangle$ because they depend on the SVD of c_{ab} .

The coefficients $\{\lambda_i\}$ are known as the Schmidt numbers, and $\#\lambda's$ is known as the Schmidt rank.

If the Schmidt rank = 1 the state is separable

$$|\Psi_{AB}\rangle = |u_1\rangle_A \otimes |v_1\rangle_B$$

If the Schmidt rank > 1 the state is entangled. More precisely, we see the marginals

$$\hat{\rho}_A = \text{Tr}_B \left(\sum_{i,i'} \lambda_i \lambda_i^* |u_i\rangle_A \langle u_{i'}| \otimes |v_i\rangle_B \langle v_{i'}| \right)$$

$$\Rightarrow \hat{\rho}_A = \sum_{i=1}^{\text{Schmidt rank}} |\lambda_i|^2 |u_i\rangle_A \langle u_i|$$

and similarly

$$\hat{\rho}_B = \sum_{i=1}^{\text{Schmidt rank}} |\lambda_i|^2 |v_i\rangle_B \langle v_i|$$

since $\langle v_i | v_{i'} \rangle = \delta_{ii'}$

$\langle u_i | u_{i'} \rangle = \delta_{ii'}$

Thus we see the interpretation of the Schmidt decomp. the Schmidt bases $\{|u_i\rangle_A\}$ $\{|v_i\rangle_B\}$ are

the eigenvectors of the respective marginals $\hat{\rho}_A, \hat{\rho}_B$ for the non-zero eigenvalues. The squares of the Schmidt numbers are the

common eigenvalues of $\hat{\rho}_A$ and $\hat{\rho}_B$ (positive #'s)

Note, through appropriate choice of phase of the Schmidt bases, we can choose the λ 's to be real

(Next page)

thus, we often write the Schmidt decomposition

$$|\Psi_{AB}\rangle = \sum_i \sqrt{p_i} |u_i\rangle_A \otimes |v_i\rangle_B$$

$$\Rightarrow \hat{\rho}_A = \sum_{i=1}^{\text{rank}} p_i |u_i\rangle_A \langle u_i| \quad \hat{\rho}_B = \sum_{i=1}^{\text{rank}} p_i |v_i\rangle_B \langle v_i|$$

The entanglement of the pure bipartite state is then seen in the von Neumann entropy of $\hat{\rho}_A$ and $\hat{\rho}_B$

$$\begin{aligned} E[|\Psi_{AB}\rangle] &= S(\hat{\rho}_A) = S(\hat{\rho}_B) \\ &= - \sum_{i=1}^{\text{Schmidt rank}} p_i \log p_i \end{aligned}$$