Lecture 21: Phase space representation for the damped S.H.O: Fokker-Planck Eq.

We have so far analyzed the master equation for the damped S.H.O. in the basis of free energy eigenstates (number states). We get further insight using a coherent state basis. Such representation connects the quantum problem to the classical problem of diffusion and Brownian motion.

Consider then the Master equation for the damped S.H.O in the absence of a coherent drive and in the interaction picture

$$\frac{d\hat{\rho}}{dt} = \frac{-\Gamma}{2}(\bar{n}+1)\left\{\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right\}$$
$$- \frac{\Gamma}{2}\bar{n}\left\{\hat{a}\hat{a}^\dagger\hat{\rho} + \hat{\rho}\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\hat{\rho}\hat{a}\right\}$$

where $\bar{n} = (e^{\frac{\hbar\omega}{kT}} - 1)^{-1}$ is the mean excitation of the thermal bath

Let us suppose $\hat{\rho}$ has a Glauber P-representation

$$\hat{\rho}(t) = \int d^2\alpha \; P(\alpha, t) \; |\alpha\rangle\langle\alpha|$$

Plugging this in we can derive an equation of motion for $P(\alpha, t)$

$$\int d^2\alpha \frac{\partial P}{\partial t} |\alpha\rangle\langle\alpha| = -\frac{\Gamma}{2}(\bar{n}+1)\int d^2\alpha\left(\{\hat{a}^\dagger\hat{a}, |\alpha\rangle\langle\alpha|\} - 2\hat{a}|\alpha\rangle\langle\alpha|\hat{a}^\dagger\right)P(\alpha)$$
$$-\frac{\Gamma}{2}\bar{n}\int d^2\alpha\left(\{\hat{a}\hat{a}^\dagger, |\alpha\rangle\langle\alpha|\} - 2\hat{a}^\dagger|\alpha\rangle\langle\alpha|\hat{a}\right)P(\alpha)$$

Aside: In order to evaluate the operator expressions we need:

$$\hat{a}|\alpha\rangle\langle\alpha| = \alpha|\alpha\rangle\langle\alpha| \quad , \quad |\alpha\rangle\langle\alpha|\hat{a}^\dagger = \alpha^*|\alpha\rangle\langle\alpha|$$

$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = ? \qquad |\alpha\rangle\langle\alpha|\hat{a} = ?$$

Now $|\alpha\rangle\langle\alpha| = e^{-|\alpha|^2} e^{\alpha a^\dagger}|0\rangle\langle 0| e^{-\alpha^* \hat{a}}$

$$\Rightarrow \hat{a}^\dagger|\alpha\rangle\langle\alpha| = e^{-|\alpha|^2}\frac{\partial}{\partial\alpha}\left(e^{\alpha a^\dagger}|0\rangle\langle 0| e^{-\alpha^*\hat{a}}\right)$$

$$= \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|$$

$$|\alpha\rangle\langle\alpha|\hat{a} = e^{-|\alpha|^2}\frac{\partial}{\partial\alpha^*}\left(e^{\alpha a^\dagger}|0\rangle\langle 0| e^{-\alpha^*\hat{a}}\right)$$

$$= \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|$$

$$\Rightarrow \{a^\dagger a, |\alpha\rangle\langle\alpha|\} - 2a|\alpha\rangle\langle\alpha|a^\dagger$$

$$= \left[\alpha\left(\frac{\partial}{\partial\alpha} + \alpha^*\right) + \alpha^*\left(\frac{\partial}{\partial\alpha^*} + \alpha\right) - 2|\alpha|^2\right]|\alpha\rangle\langle\alpha|$$

$$= \left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|$$

And $\{aa^\dagger, |\alpha\rangle\langle\alpha|\} - 2a^\dagger|\alpha\rangle\langle\alpha|\hat{a}$

$$= \left[-\left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*}\right) + 2\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]|\alpha\rangle\langle\alpha|$$

Using these and plugging in, integration by parts yields (Next Page)

$$\int \dot{P}(\alpha,\alpha^*,t)\,|\alpha\rangle\langle\alpha|\,d^2\alpha$$

$$= \frac{\Gamma}{2}\int d^2\alpha \underbrace{|\alpha\rangle\langle\alpha|}\left[\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^* + 2\bar{n}\,\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\right)\right]P(\alpha,\alpha^*,t)$$

$\Rightarrow$ Fokker–Planck Eq for quasiprobability

$$\boxed{\frac{\partial}{\partial t}P(\alpha,\alpha^*,t) = \frac{\Gamma}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)P(\alpha,\alpha^*,t) + \Gamma\bar{n}\,\frac{\partial^2 P}{\partial\alpha\,\partial\alpha^*}}$$

Change variables to quadratures: $\alpha = x + ip$

$$x = \frac{\alpha+\alpha^*}{2} \qquad p = \frac{\alpha-\alpha^*}{2i}$$

$$\Rightarrow \frac{\partial}{\partial\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right) \qquad \frac{\partial^2}{\partial\alpha\,\partial\alpha^*} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}P(x,p,t) = \frac{\Gamma}{2}\left(\frac{\partial}{\partial x}(xP) + \frac{\partial}{\partial p}(pP)\right) + \frac{\Gamma\bar{n}}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right)P(x,p,t)}$$

This equation is known in classical stochastic theory as the Fokker–Planck equation

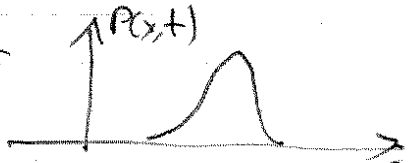
General form of F.P eq.

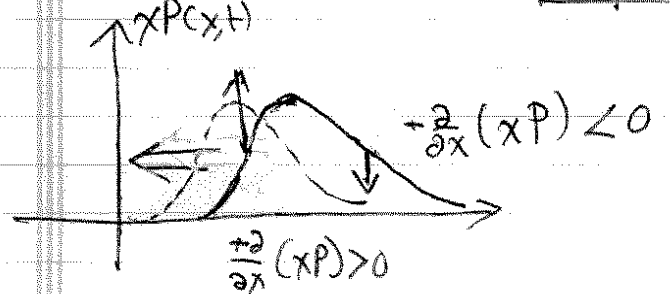
One random variable x

$$\frac{\partial P}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(M_1 x P(x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(M_2 P(x,t)\right)$$

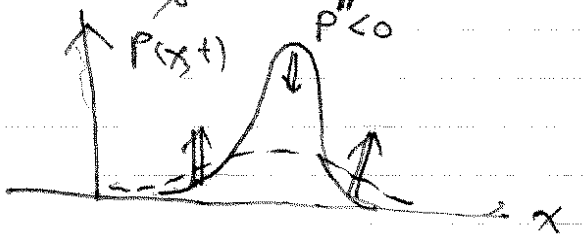
M_1: Drift coefficient; M_2: Diffusion coeff.

Suppose at some time t





Drift



Diffusion

Effects made clear by considering

$$\frac{d}{dt}\langle x\rangle = \int dx\, x \frac{\partial P}{\partial t} = \int dx\, x\left(+\frac{\partial}{\partial x}(M_1 x P) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(M_2 P)\right)$$

$$= -M_1 \int dx\, xP = -M_1\langle x\rangle = -\frac{\Gamma}{2}\langle x\rangle$$

(exponential decay at rate $M_1 = \frac{\Gamma}{2}$)

$$\frac{d}{dt}\langle \Delta x^2\rangle = \frac{d}{dt}\langle x^2\rangle - 2\langle x\rangle \frac{d}{dt}\langle x\rangle$$

$$= -2M_1\langle x^2\rangle + M_2 + 2M_1\langle x\rangle^2 = M_2$$

$$= -\Gamma\left(\langle x^2\rangle - \langle x\rangle^2\right) + M_2 = -\Gamma\langle \Delta x^2\rangle + M_2$$

$\Rightarrow$ Solution! $\langle \Delta x^2\rangle_t = \langle \Delta x^2\rangle_0 e^{-\Gamma t} + \left(1 - e^{-\Gamma t}\right)\frac{M_2}{\Gamma}$

Damped oscillator: $M_1 = \frac{\Gamma}{2}$ $\quad M_2 = \frac{\Gamma \bar{n}}{2}$

"Fluctuation dissipation theorem"

Diffusion coeff related to decay rate

$$\frac{d}{dt}\langle \alpha \rangle = -\frac{\Gamma}{2}\langle \alpha \rangle$$

$$\langle \Delta |\alpha|^2 \rangle_{S.S.} = \langle \Delta x^2 \rangle + \langle \Delta p^2 \rangle = \bar{n} \quad \text{thermal state}$$

In fact this linear P.D.E. can be solved in special case

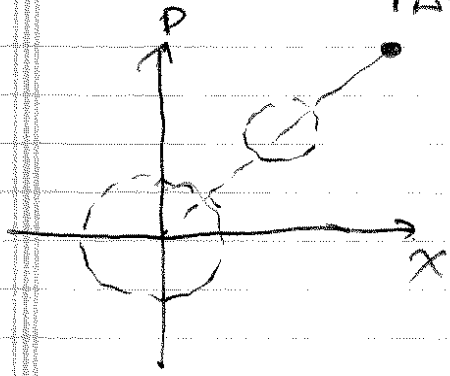
Suppose we are initially in a coherent state $|\alpha_0\rangle$

$$\Rightarrow P(\alpha, \alpha^*, 0) = \delta(\alpha - \alpha_0)$$

General solution $P(\alpha, \alpha^*, t) = \frac{1}{|\Delta\alpha(t)|^2} e^{-\frac{|\alpha - \alpha(t)|^2}{|\Delta\alpha(t)|^2}}$

where $\alpha(t) = \alpha_0 e^{-\frac{\Gamma}{2}t}$

$$|\Delta\alpha(t)|^2 = \bar{n}(1 - e^{-\Gamma t})$$



Decay and diffusion

Steady-state $t \to \infty$

$$P_{S.S.}(\alpha, \alpha^*) = \frac{1}{\bar{n}} e^{-\frac{|\alpha|^2}{\bar{n}}}$$

Thermal equilibrium with reservoir

Note: If $T = 0$ (vacuum) $P(\alpha, t)$ remains a coherent state for all t, with decaying amplitude

Damping of a squeezed coherent state

Suppose at $t=0$ $\quad |\psi(0)\rangle = \hat{D}(\alpha)\,\hat{S}(r)\,|0\rangle$

This state does not have a Glauber P-representation.

However we can instead look at the, e.g., Wigner fct.

The same kind of Fokker–Planck equation holds, but with a larger diffusion coeff. $\quad \bar{n} \Rightarrow \bar{n} + \frac{1}{2}$

$$\frac{\partial}{\partial t} W(x,p,t) = \frac{\Gamma}{2}\left(\frac{\partial}{\partial x}\,x + \frac{\partial}{\partial p}\,p\right) W(x,p,t)$$
$$+ \frac{\Gamma}{4}\left(\bar{n}+\frac{1}{2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right) W(x,p,t)$$

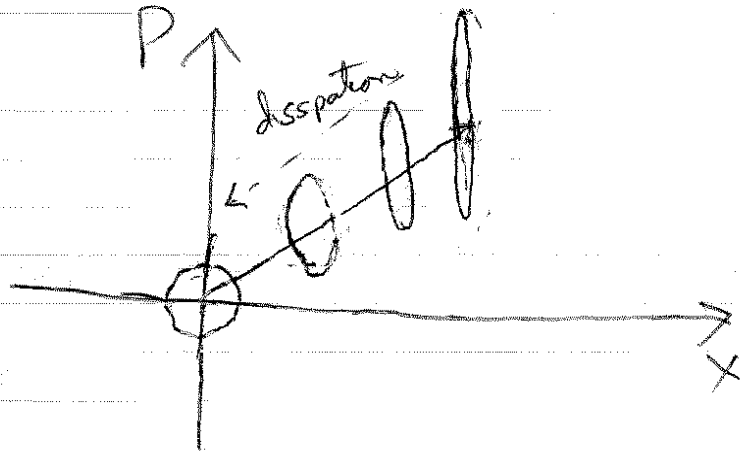
The FP equation maps Gaussians into Gaussians

$\Rightarrow$ Ansatz: $\quad W(x,p,t) = \dfrac{1}{2\pi\,\Delta x(t)\,\Delta p(t)}\; e^{\frac{-x^2}{2\Delta x(t)^2}}\; e^{\frac{-p^2}{2\Delta p(t)^2}}$

Solution:

$$\Delta x^2(t) = \Delta x^2(0)\,e^{-\Gamma t} + \left(1-e^{-\Gamma t}\right)\left(\frac{\bar{n}+\frac{1}{2}}{2}\right)$$

$$\Delta p^2(t) = \Delta p^2(0)\,e^{-\Gamma t} + \left(1-e^{-\Gamma t}\right)\left(\frac{\bar{n}+\frac{1}{2}}{2}\right)$$



e.g. $\bar{n} = 0$

Damping kills Squeezing

Damping vs. Decoherence

So far our examples have shown that the master equation for the damped SHO have lead to damping and diffusion. These are a "classical" phenomena; familiar in stochastic processes (e.g. Brownian motion). The coupling of a quantum system to an enviroment (reservoir) also leads to very nonclassical effects.

Decoherence of a "Schrödinger Cat"

Suppose we start at $t=0$ with a "macroscopic superposition" of two coherent states

$$|\psi(0)\rangle = \sqrt{\frac{C}{2}}\left(|\alpha_0\rangle + |-\alpha_0\rangle\right)$$

E.g. α_0 real

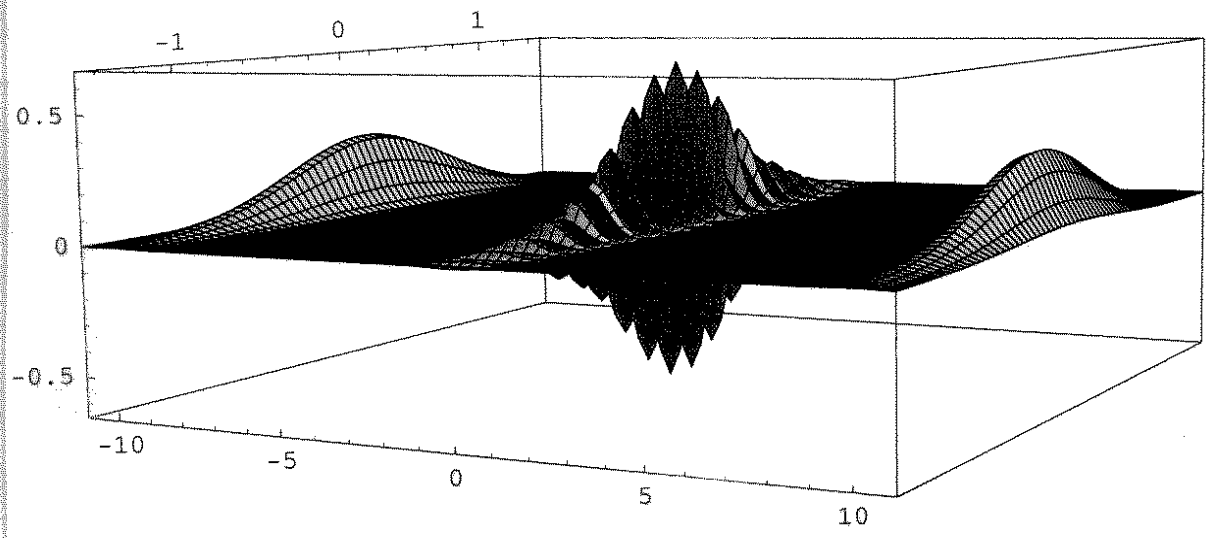
where $\alpha_0 = X_0 = \left(\frac{X_0}{\sqrt{\frac{\hbar}{2m\omega_0}}}\right)$ For mechanical oscillator

Macroscopic $\Rightarrow$ $X_0 \gg 1$

We found the Wigner function in the Problem Set

$$W(X,P,0) = \frac{C}{2}\left[W_+(x,p) + W_-(x,p) + \frac{4}{\pi}\cos(4X_0 p)\, e^{-2(x^2+p^2)}\right]$$

Where $W_\pm(x,p) = \frac{2}{\pi} e^{-2\{(x-X_0)^2 + p^2\}}$



The Wigner function shows localized Gaussians at $x = \pm x_0$, $p = 0$. In addition it shows rapid oscillations in p at a rate depending on x_0. These come from the "coherence", that is a statistical mixture of coherent states

$$\hat{\rho} = C\left(\frac{1}{2}|\alpha_0\rangle\langle\alpha_0| + \frac{1}{2}|-\alpha_0\rangle\langle-\alpha_0|\right)$$

$$\Rightarrow W(x,p) = C\left(\frac{W_+(x,p)}{2} + \frac{W_-(x,p)}{2}\right)$$

The "interference in phase space" distinguishes the statistical mixture from the "Schrödinger cat" which is in a superposition of two macroscopic possibilities

However, diffusion in the FP equation very quickly wipes out the coherence for macroscopic superpositions

The oscillatory term $\cos(2pX_0)$ under diffusion in p:

$$\Gamma\left(\bar{n}+\tfrac{1}{2}\right)\frac{\partial^2}{\partial p^2}\cos(2pX_0) = \Gamma\left(\bar{n}+\tfrac{1}{2}\right)X_0^2\cos(2pX_0)$$

$\Rightarrow$ Amplitude of oscillation decay at rate

$$\gamma_{coh} = X_0^2\,\bar{n}\,\Gamma = \Gamma\left(\frac{X_0^2}{\hbar/2m\omega}\right)\bar{n}$$

In high temperature limit $\bar{n} = \dfrac{kT}{\hbar\omega}$

$$\Rightarrow \gamma_{coh} = \Gamma\left(\frac{X_0^2}{\frac{\hbar^2}{2mkT}}\right) = \Gamma\left(\frac{X_0}{\lambda_{DB}}\right)^2$$

where $\lambda_{DB} = \dfrac{\hbar}{\sqrt{2mkT}}$ is the thermal deBroglie wavelength

The lifetime of coherence

$$\tau_{coh} = \frac{1}{\gamma_{coh}} = \tau_{decay}\left(\frac{\lambda_{DB}}{X_0}\right)^2 \qquad \text{where } \tau_{decay} = \frac{1}{\Gamma}$$

For example, at room temperature, $m = 1$ gram, $X_0 = 1$ cm

$$\Rightarrow \frac{\tau_{coh}}{\tau_{decay}} = 10^{-40}$$

Even if $\tau_{decay} = 10^{17}$ s (age of universe)

$\tau_{coh} \sim 10^{-23}$ s