

Lecture 26: Different Unravelings of the Master Eqn.

We have seen that the (non-unitary) evolution of the density operator under a "Lindblad" master equation can be interpreted as an average over "quantum trajectories" - stochastically evolving wave functions. This can be thought of as the environment performing a measurement of the system, but not telling us the result. This ~~no~~ evolution generally turns pure states into mixed states.

The diagram illustrates the decomposition of the density operator $\hat{\rho}(t)$ into a sum of projectors. On the right, the expression $\hat{\rho}(t) = \lim_{N \rightarrow \infty} \sum_i \frac{|\psi_i(t)\rangle\langle\psi_i(t)|}{N}$ is shown. To its left, a large circle represents the environment's state. Inside the circle, three projectors are labeled: $|\psi_1(t)\rangle\langle\psi_1(t)|$ at the top, $|\psi_2(t)\rangle\langle\psi_2(t)|$ in the middle, and $|\psi_N(t)\rangle\langle\psi_N(t)|$ at the bottom. Arrows point from each projector inside the circle to the corresponding term in the summation formula on the right.

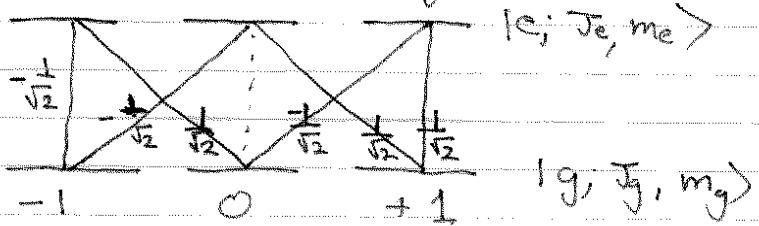
This is the evolution had the environment performed such a measurement, but of course, there is ~~not~~ no such real measurement. In general we can imagine a different measurement made on the environment. This will lead to a different set of quantum trajectories, though the average must lead to the same density since this was an imagined alternative measurement!

These alternative sets of quantum trajectories are known as different unravelings of the master equation. It will be our goal to understand how these are related and how they fit into our modern knowledge of quantum measurement theory.

In this lecture I will begin with some basic examples to motivate how different unravellings give different pictures of quantum trajectories.

Example: Dark states and coherent population trapping

Consider a multilevel atom with Zeeman degeneracy. In particular consider an atom whose ground and excited state have total angular momentum $J_e = J_g = 1$.



States $|e; J_e, m_e\rangle$ and $|g; J_g, m_g\rangle$ are connected by electric dipole selection rules

if $m_e - m_g = q = 0, \pm 1$. These transitions are driven by photons of polarization \vec{e}_q

$$\vec{e}_0 = \vec{e}_z, \quad \vec{e}_{\pm 1} = \mp \left(\frac{\vec{e}_x \pm i\vec{e}_y}{\sqrt{2}} \right) \text{ w.r.t. to the space fixed}$$

quantization axis \vec{e}_z for which the atoms angular momentum eigenstates are defined.

The dipole matrix elements are determined by the Wigner-Eckart theorem

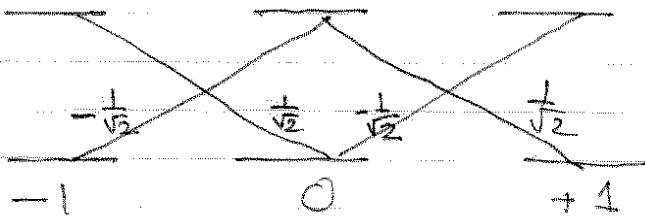
$$\langle e; J_e m_e | \hat{d}_q | g; J_g m_g \rangle = \underbrace{\langle J_e m_e | J_g m_g | g \rangle}_{\text{Clebsch-Gordan coeff}} \underbrace{\langle \hat{d} | \hat{d}_q \rangle}_{m_e = m_g + q} \text{ Reduced matrix elem.}$$

$$\hat{d}_q = \vec{e}_q \cdot \hat{d}$$

The Clebsch-Gordan coefficients for our atom are shown above. Two important points should be noted.

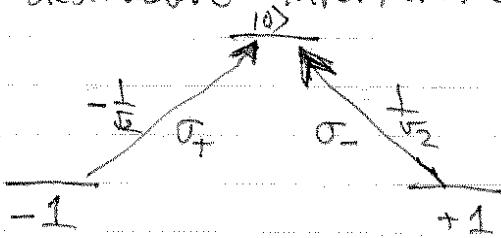
- (i) The CG coeff for $|J_g m_g=0\rangle \rightarrow |J_e = m_e=0\rangle$ is zero
- (ii) Some CG coeff are positive and others are negative.

Consider an atom driven by linearly polarized light. If we take the quantization axis along the wave vector $\hat{k} \parallel \hat{z}$, the light is seen as a superposition of polarized (no π -transition)



The state $|1\psi_{\text{dark}}\rangle = (|g; -1\rangle + |g, +1\rangle)/\sqrt{2}$

is a dark state; it is uncoupled from the laser due to destructive interference.



$$\text{Let } \hat{d} \cdot \hat{e}_z |\psi_{\text{dark}}\rangle = 0$$

This is a "Lambda" system of the sort studied in Problem Set #3.

If an atom ~~not~~ starts in a arbitrary initial state it will, through absorption and emission, eventually end up in the dark state. This process is known in laser spectroscopy as "coherent population trapping".

That is, in steady state $\hat{\rho}_{\text{ss}} \rightarrow |1\psi_{\text{dark}}\rangle \langle 1\psi_{\text{dark}}|$

Our goal is to see how different unravellings can lead to the same steady state.

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The Master equation for the Multilevel atom

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \Gamma \sum_q \hat{D}_q \hat{\rho} \hat{D}_q^+ \quad \text{Lindblad form}$$

$$\hat{H}_{\text{eff}} = \hat{H} - \frac{\hbar\Gamma}{2} \sum_q \hat{S}_q^+ \hat{S}_q^-$$

$$\hat{D}_q^+ = \sum_{m_g} \langle J_e m_g | J_g m_g | 1_g \rangle \quad | 1_g m_g \rangle \langle J_e m_g |$$

"Jump operators" emission of photon \vec{e}_g
 $\sum_{m_g} (\text{out } | 1_g m_g \rangle) \rightarrow | J_g m_g \rangle$

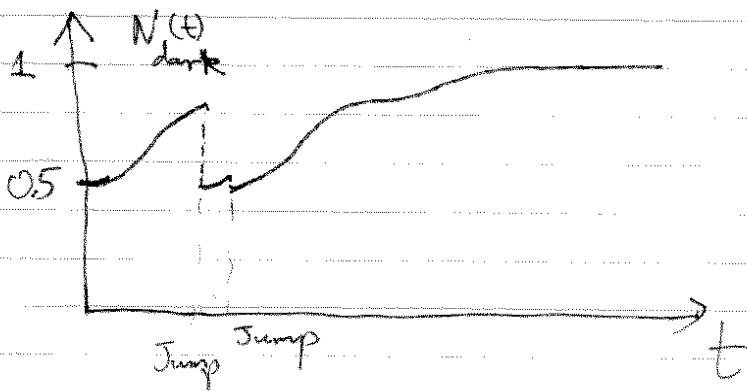
$$\sum \hat{D}_q^+ \hat{D}_q^- = \hat{P}_e = \sum_{m_e} | J_e m_e \rangle \langle J_e m_e |$$

In rotating frame $\hat{H} = -\frac{\hbar D}{2} (\hat{J} \cdot \vec{E}_L + \hat{J}^+ \cdot \vec{E}_L^*)$

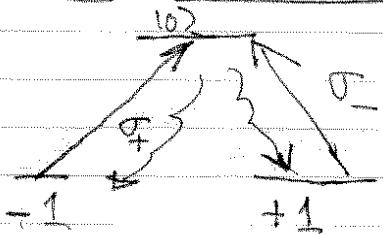
$$\vec{D} = \sum_q \hat{D}_q \vec{e}_g^*$$

Let us examine the unravelling of the master equation for this set of jump operators. Physically this corresponds to detectors which measure photons with three possible polarizations $\vec{e}_g = \vec{e}_2, \vec{e}_+, \vec{e}_-$. This would require tricky optics, but possible.

Suppose at $t=0$ $| \Psi(0) \rangle = | g; m=-1 \rangle$. Let us plot $| \langle \Psi_{\text{dark}} | \Psi(t) \rangle |^2 = N_{\text{dark}}(t)$ for some trajet.



As expected, in steady state the population is trapped in the dark state. However, it reached this state through a long series of null measurements



Because $|0\rangle \rightarrow |0\rangle$ is forbidden, jumps put the atom into $|m_g\rangle = |\pm 1\rangle$

However two important states

$$|\psi_{\text{Dark}}\rangle = \frac{|1\rangle + |1\rangle}{\sqrt{2}}$$

$$|\psi_{\text{Bright}}\rangle = \frac{|1\rangle - |1\rangle}{\sqrt{2}}$$

In the absence of a detector

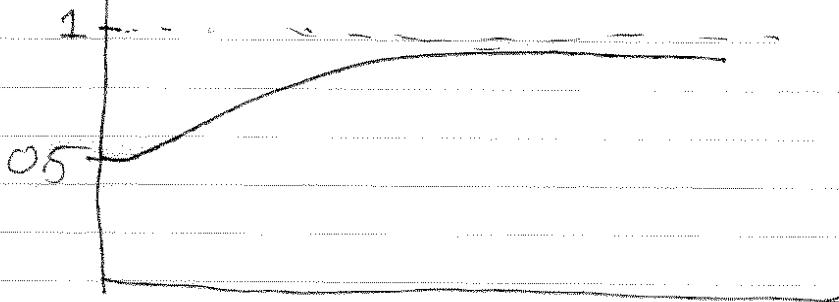
$$|\psi(t)\rangle = C_D(t) |\psi_{\text{Dark}}\rangle + C_B(t) |\psi_{\text{Bright}}\rangle$$

As time increases $C_B(t)$ decays

$C_D(t)$ rises and asymptotes.

Thus, in this unravelling, evolution to the dark state is seen as a continuous rotation of the state. After averaging over many trajectories,

$$N_{\text{Dark}}(t) = \langle \psi_{\text{dark}} | \hat{\rho}(t) | \psi_{\text{dark}} \rangle$$

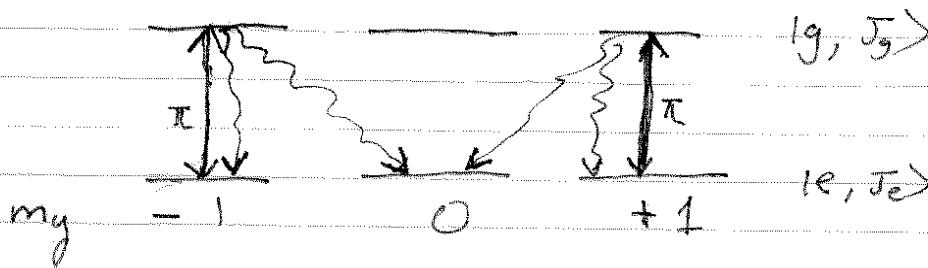


Now let us consider an alternative unravelling.

Suppose our detectors are sensitive to the photon polarizations

$$\vec{e}_y, -\left(\frac{\vec{e}_z + i\vec{e}_x}{\sqrt{2}}\right), \left(\frac{\vec{e}_z - i\vec{e}_x}{\sqrt{2}}\right)$$

This corresponds to measuring photon angular momentum with \vec{e}_y as the quantization axis. Choosing this axis to be the direction of laser polarization the driven transitions are:



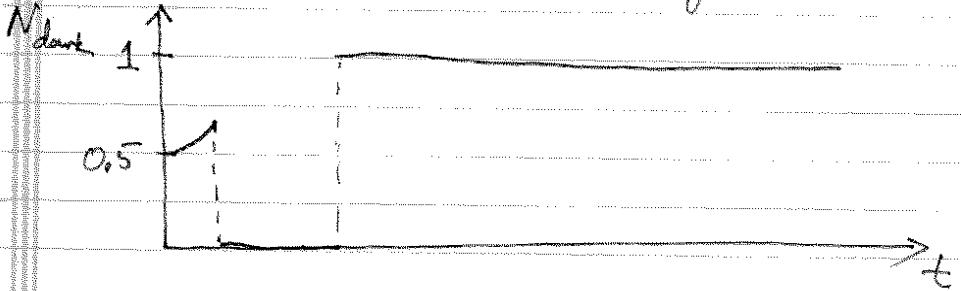
With respect to this quantization axis there is only π -polarized light. Spontaneous transitions are also drawn above.

Here $|g, J_g, m_g = 0\rangle$ is a dark state. Of course this is the same state $|\Psi_{\text{dark}}\rangle$, just written in a different basis.

$$|\Psi_{\text{dark}}\rangle = \frac{1}{\sqrt{2}} (|g, J_g, m_g = 1\rangle + |g, J_g, m_g = -1\rangle) = |g, J_g, m_g = 0\rangle$$

How do quantum trajectories look w.r.t. this set of jumps?

Plotting $N_{\text{dark}}(t) = |\langle \hat{\psi}_{\text{dark}} | \hat{\psi}(t) \rangle|^2$ with $|\hat{\psi}(0)\rangle = |m_z = -1\rangle$ for a sample trajectory

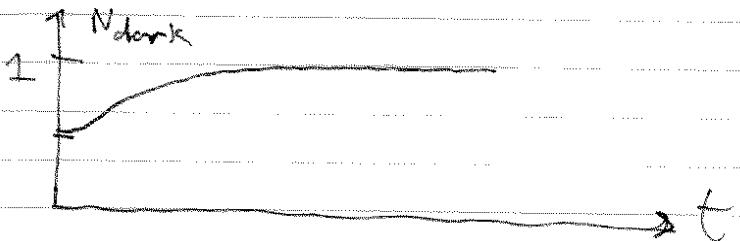


$$\text{At } t=0 \quad |\hat{\psi}(0)\rangle = |m_z = -1\rangle = \frac{1}{2}(|m_y = -1\rangle + |m_y = 1\rangle) + \frac{1}{\sqrt{2}}|m_y = 0\rangle$$

thus, during the time $t=0 \rightarrow t$ of first jump the state continuously rotates towards the dark state $|\hat{\psi}_{\text{dark}}\rangle = |m_y = 0\rangle$.

After the first jump, the atom is projected onto $|m_y = -1\rangle$, $|m_y = 0\rangle$, or $|m_y = +1\rangle$. For $|m_y = \pm 1\rangle$ there is no dark state component. If the atom, after a jump is projected onto $|m_y = 0\rangle$ it lands in the dark state and remains there ever after! This is the case for the second jump shown in the sample trajectory above.

* Of course, in the average over many trajectories we must recover the expectation value of the density operator



In this unravelling we would need many more trajectories to yield the expectation value of $\hat{\rho}$ as compared with the first unravelling we considered. This is because the projector onto the dark state involves an eigenstate of one of the jump operators.

Thus, the same Master eqn (in Lindblad form)

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \hat{L}_{\text{relax}}[\hat{\rho}]$$

is generated from an equivalence class of jump operators if they are related by

$$\hat{D}_m = \hat{T}^\dagger \hat{C}_m \hat{T}$$

for unitary transformations \hat{T} such that

$$\hat{T}^\dagger \hat{L}_{\text{relax}}[\hat{\rho}] \hat{T}^\dagger = \hat{L}_{\text{relax}}[\hat{T}^\dagger \hat{\rho} \hat{T}^\dagger].$$

For example, suppose we can consider relaxation by coupling the atom to the vacuum. ~~Now~~ the vacuum is isotropic. Thus rotating the angular momentum of the atom does not affect the relaxation process. Therefore, there is an equivalence class of jump operators, which correspond to measurements of the photon angular momentum along different quantization axes, which yield the same master eqn.

The equivalence class we have given here is not the whole story. To find all possible equivalent unravellings we need to consider all possible preparations of the system that can be achieved by measurement of the environment. We tackle this problem in the next lecture.

These different unravelings demonstrate that steady state behavior can be reached in very different ways: continuous evolution vs. quantum jumps.

We would like to know how these different unravelings are related, and what set of different unravelings are consistent with one another, i.e., yield the same master equation.

One sufficient set is given by looking at the invariance of $\hat{L}_{\text{relax}}[\hat{\rho}]$ under unitary transformation.

Suppose \hat{T} is a unitary operator such that

$$\hat{T} \hat{L}_{\text{relax}}[\hat{\rho}] \hat{T}^\dagger = \hat{L}_{\text{relax}}[\hat{T} \hat{\rho} \hat{T}^\dagger]$$

\Rightarrow The relaxation process is invariant under the symmetry \hat{T}

$\Rightarrow \hat{\rho}$ and $\hat{T} \hat{\rho} \hat{T}^\dagger$ satisfy the same master eqn, with respect to the rotated Hamiltonian $\hat{T} \hat{A} \hat{T}^\dagger$.

Alternatively we see that

$$\begin{aligned} \hat{L}_{\text{relax}}[\hat{\rho}] &= \hat{T}^\dagger \hat{L}_{\text{relax}}[\hat{T} \hat{\rho} \hat{T}^\dagger] \hat{T} \\ &= -\frac{1}{2} \sum_m (\hat{D}_m^\dagger \hat{D}_m \hat{\rho} + \hat{\rho} \hat{D}_m^\dagger \hat{D}_m) \\ &\quad + \sum_m \hat{D}_m \hat{\rho} \hat{D}_m^\dagger \end{aligned}$$

$$\text{where } \hat{D}_m = \hat{T}^\dagger \hat{C}_m \hat{T}$$