Lecture 28: the Stochastic Schrödinger Equation

We saw in the lecture 27, when a "system" interacts with an "environment", the dynamics of the system alone is described by a "completely positive map"

$$\hat{\rho}_{out} = \$[\hat{\rho}_{in}] = \sum_\mu \hat{M}_\mu^\dagger \hat{\rho}_{in} \hat{M}_\mu$$

where $\{\hat{M}_\mu\}$ define a Krause "operator-sum" representation.

We interpret this as an ensemble decomposition

$$\hat{\rho}_{out} = \sum_\mu p_\mu \left.\hat{\rho}_{out}\right|_\mu$$

where $p_\mu = \mathrm{Tr}(\hat{M}_\mu^\dagger \hat{M}_\mu \hat{\rho}_{in})$ = Probability of measuring μ with POVM $\hat{M}_\mu^\dagger \hat{M}_\mu$

$$\left.\hat{\rho}_{out}\right|_\mu = \frac{\hat{M}_\mu \hat{\rho}_{in} \hat{M}_\mu^\dagger}{\mathrm{Tr}(\hat{M}_\mu \hat{\rho}_{in} \hat{M}_\mu^\dagger)} = \text{Output state conditioned on measurement } \mu$$

Since the ensemble decomposition is not unique, neither is the Krause decomposition. The same superoperator is determined by a "unitary remixing"

$$\hat{K}_\nu = \sum_\mu U_{\nu\mu} \hat{M}_\mu$$

The Linblad equation as a Complete positive map

The master equation (Linblad form) was derived fundamentally from the bipartite "system-reservoir" model. Thus, the solution is described by a completely positive map

$$\hat{\rho}(t) = \$_t[\hat{\rho}(0)]$$

In the Markov approximation this takes the form of a differential equation, generated by the "Linbladian"

$$\hat{\rho}(t) = e^{\mathcal{L}t}[\rho(0)]$$

$$d\hat{\rho}(t) = \mathcal{L}[\hat{\rho}(t)]\,dt$$

$$\mathcal{L}[\hat{\rho}] = -\frac{i}{\hbar}[\hat{H}_{eff}\hat{\rho}]' + \sum_{\mu=1}^{N} \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$$

$$\hat{H}_{eff} = \hat{H} - \frac{i}{2}\sum_{\mu=1}^{N} \hat{L}_\mu^\dagger \hat{L}_\mu$$

The fact that the dynamics of $\hat{\rho}(t)$ depends only on t and not the whole history is the essence of the Markov approx.

The differential solution is a completely positive map

$$\hat{\rho}(t+dt) = \$_{dt}[\hat{\rho}(t)] = \hat{\rho}(t) + \mathcal{L}[\hat{\rho}(t)]\,dt$$
$$= \sum_{\mu=0}^{N} \hat{M}_\mu(dt)\,\hat{\rho}(t)\,\hat{M}_\mu^\dagger(dt)$$

$\mu > 1 \quad \hat{M}_\mu = \hat{L}_\mu \sqrt{dt}$

$\mu = 0 \quad \hat{M}_0 = \hat{\mathbb{1}} + \underbrace{\left(-\frac{i}{\hbar}\hat{H} - \frac{1}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu\right)}_{-\frac{i}{\hbar}\hat{H}_{eff}} dt$

To order $O(dt)$ $\hat{M}_0^\dagger \hat{M}_0 + \sum_{\mu=1}^{N} \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{\mathbb{1}}$ as is must

This set of Krause operators corresponds to a set of POVMS $\hat{E}_\mu = \hat{M}_\mu^\dagger \hat{M}_\mu$ and an ensemble decomposition. An "unravelling" corresponds to a stochastically evolving state relative to this particular measurent records.

Given a pure state at time t, and a measurement value μ

$$|\psi(t+dt)\rangle = \frac{\hat{M}_\mu |\psi(t)\rangle}{\| \hat{M}_\mu |\psi(t)\rangle \|}$$ with probability $P_\mu = \langle\psi(t)|\hat{M}_\mu^\dagger \hat{M}_\mu|\psi(t)\rangle$

This stochastically evolving state is our "quantum trajectory"

We can formalize this using "stochastic calculus".

In unnormalized form

$$|\tilde{\psi}(t+dt)\rangle = \sum_{\mu=0}^{m} dN_\mu(t)\ \hat{M}_\mu\, |\tilde{\psi}(t)\rangle$$

where $dN_\mu(t)$ is a stochastic inverval taking the random values

$$dN_\mu(t) = \begin{cases} 1 & \text{with prob } p_\mu = \dfrac{\langle\tilde{\psi}(t)|\hat{M}_\mu^\dagger \hat{M}_\mu|\tilde{\psi}(t)\rangle}{\|\tilde{\psi}(t)\|^2} \\ 0 & \text{with prob } 1-p_\mu \end{cases}$$

$$\Rightarrow \langle dN_\mu \rangle = p_\mu \qquad \sum_{\mu=0}^{m} dN_\mu = 1$$

($\langle dN_\mu \rangle$: average of)

Separating the "no jump" from "jump"

$$|\tilde{\psi}(t+dt)\rangle = dN_0 \left(\hat{\mathbb{1}} + \frac{-i}{\hbar}\hat{H}_{\text{eff}}\, dt\right)|\tilde{\psi}(t)\rangle + \sum_{\mu=1}^{m} dN_\mu\, \hat{L}_\mu \sqrt{dt}\, |\tilde{\psi}(t)\rangle$$

Now $dN_0 = 1 - \sum_{\mu=1}^{m} dN_\mu$

$$\Rightarrow |\tilde{\psi}(t+dt)\rangle = \left(\hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}_{\text{eff}}\, dt\right)|\tilde{\psi}(t)\rangle + \sum_{\mu=1}^{m} dN_\mu \left(\hat{L}_\mu \sqrt{dt} - \hat{\mathbb{1}}\right)|\tilde{\psi}(t)\rangle$$

where I used $dN_\mu\, dt = 0$ since dN_μ is now zero on a set of measure zero

We thus arrive at the "Stochastic Schrödinger equation"

$$d|\tilde{\psi}(t)\rangle = |\tilde{\psi}(t+dt)\rangle - |\tilde{\psi}(t)\rangle$$

$$\Rightarrow \boxed{d|\tilde{\psi}(t)\rangle = \left[-\frac{i}{\hbar}\hat{H}_{eff}\,dt + \sum_{\mu=1}^{m} dN_\mu\left(\hat{L}_\mu\sqrt{dt} - 1\right)\right]|\tilde{\psi}(t)\rangle}$$

Note, though this equation looks linear it is not since $dN_\mu(t)$ depends on $|\tilde{\psi}(t)\rangle$.

We can see this explicitly by writing this equation for the normalized state $|\psi(t)\rangle \equiv \dfrac{|\tilde{\psi}(t)\rangle}{\sqrt{\langle\tilde{\psi}(t)|\tilde{\psi}(t)\rangle}}$

If $d|\tilde{\psi}\rangle = \hat{G}_{dH}|\tilde{\psi}\rangle \Rightarrow d|\psi\rangle = \left(\hat{G} - \dfrac{\langle\hat{G}+\hat{G}^\dagger\rangle}{2}\right)_{dH}|\psi\rangle$

$d|\tilde{\psi}\rangle = \sum_\mu dN_\mu\left(\hat{G}-1\right)|\tilde{\psi}\rangle \Rightarrow d|\psi\rangle = \left(\dfrac{\hat{G}}{\sqrt{\hat{G}^\dagger\hat{G}}} - 1\right)dN\,|\psi\rangle$

$$\therefore \boxed{d|\psi\rangle = \left[\left(-\frac{i}{\hbar}\hat{H} - \frac{1}{2}\sum_\mu\left(\hat{L}_\mu^\dagger\hat{L}_\mu - \langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle\right)\right)dt + \sum_{\mu=1}^{m} dN_\mu(t)\left(\frac{\hat{L}_\mu}{\sqrt{\langle\hat{L}_\mu^\dagger\hat{L}_\mu\rangle}} - 1\right)\right]|\psi\rangle}$$

Different unravellings $\Leftrightarrow$ Unitary remixing of Kraus operators

All operator sum decompositions are related by

$$\$[\hat{\rho}] = \sum_{\mu=0}^{m} \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger = \sum_{\nu=0}^{k} \hat{K}_\nu \hat{\rho} \hat{K}_\nu^\dagger$$

iff $\hat{K}_\nu = \sum_{\mu=0}^{m} U_{\nu\mu} \hat{M}_\mu$ where $\{U_{\nu\mu}\}$ are the $k+1$ columns of a unitary matrix.

When $\$$ is the differential map of the master equation, these unitary remixings define different unravellings.

We saw in lecture 26 that, given a symmetry $\hat{T}$ s.t.

$$\hat{T}\, \mathcal{L}_{\text{redux}}[\hat{\rho}]\, \hat{T}^\dagger = \mathcal{L}_{\text{redux}}[\hat{T}\hat{\rho}\hat{T}^\dagger]$$

Then $\{\hat{D}_\mu = \hat{T}^\dagger \hat{L}_\mu \hat{T}\}$ are a set of Lindblad operators that give the same master eqn.

This is an example of a unitary remixing of the Kraus ops.

We have $\hat{M}_\mu = \hat{L}_\mu \sqrt{dt} \quad \mu = 1, \ldots, m$

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Thus $\hat{T}$ is a symmetry of $\mathcal{L}_{\text{Lindblad}}$ iff

$$\hat{T}^\dagger \hat{L}_\mu \hat{T} = \sum_{\nu=1}^{m} U_{\mu\nu} \hat{L}_\nu = \hat{D}_\mu$$

That is, the new Jump operators are linear combinations of the old via unitary matrix.

For example, in Lecture 26 we saw that given jump operators $\hat{D}_q$ corresponding to emmission of a photon with ~~that~~ polarization $\vec{e}_q$ along z-axis, an alternative set is $\hat{T}^\dagger \hat{D}_q \hat{T}$ where $\hat{T}$ is a rotation operator. This new set is a linear combination of the old:

$$\hat{D}_q = \vec{e}_q \cdot \vec{\hat{D}}$$

$$\hat{T}^\dagger \hat{D}_q \hat{T} = \sum_{q'} \underbrace{(U_{qq'} \vec{e}_{q'})}_{\text{Unitary rotation}} \cdot \vec{\hat{D}} = \sum_{q'} U_{qq'} \hat{D}_{q'}$$

Unitary ~~set~~ mixing of Krause operators.