

Lecture 5b: Two-level atomic response

• "Natural Lineshape"

Recall the time evolution of mean dipole-moment ("lab frame")

$$\langle \vec{d}(t) \rangle = \vec{d}_{eg} (u(t) \cos \omega t - v(t) \sin \omega t)$$

$u(t) \Rightarrow$ In phase with $\vec{E} \Rightarrow$ Index of refraction

$-v(t) \Rightarrow$ In quadrature with $\vec{E} \Rightarrow$ Absorption / emission

$$\frac{u+iv}{2} = \rho_{ge}$$

Steady state \Rightarrow After transients decay $\Rightarrow \dot{\rho}_{s.s.} = 0$

\Rightarrow Steady state coherences $\dot{\rho}_{ge} = 0$

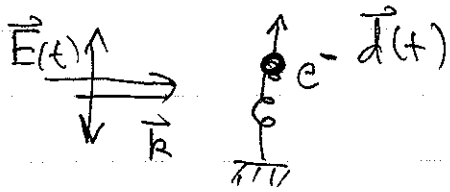
$$\Rightarrow 0 = \left(-i\Delta - \frac{\Gamma}{2}\right) \rho_{ge}^{s.s.} + i\frac{\Omega}{2} (\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.})$$

$$\Rightarrow \boxed{\rho_{ge}^{s.s.} = \left(\frac{\Omega/2}{\Delta - i\Gamma/2}\right) (\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.})}$$

Suppose in steady state $\rho_{ee}^{s.s.} \approx 0$ (We come back to see when true)
 $\Rightarrow \rho_{gg} \approx 1$

\Rightarrow For weak excitation $\rho_{ge}^{s.s.} = \left(\frac{-\Omega/2}{\Delta - i\Gamma/2}\right)$ "Complex Lorentzian"

"Linear response" \Rightarrow Classical S.H.O.



$$\vec{d}(t) = \text{Re}(\tilde{\alpha} \vec{E}_0 e^{-i\omega t})$$

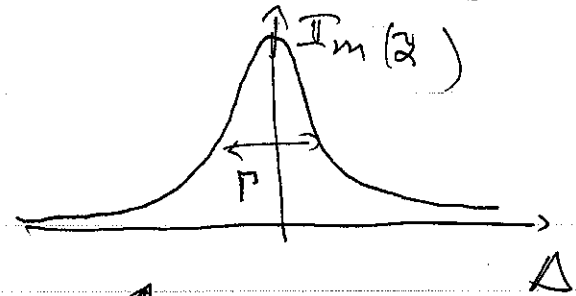
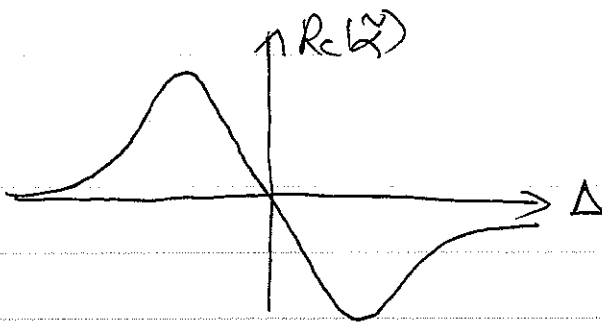
\uparrow polarizability

$$\langle \vec{d}(t) \rangle_{ss} = \text{Re} \left[\vec{d}_{eg} \underbrace{(u^{ss} - i v^{ss})}_{\alpha^{ss}} e^{-i\omega t} \right]$$

$$\alpha^{ss} = \frac{-\Omega}{\Delta + i\Gamma/2} = \frac{-\vec{d}_{eg} \cdot \vec{E}}{\hbar(\Delta + i\Gamma/2)}$$

\therefore Quantum polarizability

$$\tilde{\alpha} = \frac{-|\vec{d}_{eg}|^2}{\hbar(\Delta + i\Gamma/2)} = \frac{+|\vec{d}_{eg}|^2}{\hbar} \left(\frac{-\Delta + i\Gamma/2}{\Delta^2 + \Gamma^2/4} \right)$$



\uparrow
Lorentzian

Natural Line width = Γ

\Rightarrow Absorption strong when $\Delta < \Gamma$

Rate equations

In the regime when the transient dynamics have damped out and the coherences reach steady state, we can look at population dynamics.
Plugging in the coherences ~~from~~ is s.s.

$$\Rightarrow \dot{\rho}_{ee} = -\Gamma \rho_{ee} - \Omega \operatorname{Im}(\rho_{ge}^{\text{s.s.}})$$

$$\dot{\rho}_{ee} = -\Gamma \rho_{ee} + \left(\frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4} \right) (\rho_{ee} - \rho_{gg})$$

Interpretation: Population rate equations

$$N_e \equiv \rho_{ee}$$

$$N_g \equiv \rho_{gg}$$

$$\frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4} \equiv \Gamma_{\text{stim}} = \begin{array}{l} \text{stimulated absorption} \\ \text{stimulated emission} \end{array}$$

Check: Fermi's Golden rule:

$$\Gamma_{\text{stim}} = \frac{2\pi}{\hbar^2} |\langle e | \hat{H}_{\text{int}}^{(+)} | g \rangle|^2 \mathcal{D}(\omega_L)$$

↑
density of states

$$\hat{H}_{\text{int}}^{(+)} = -\frac{\hbar\Omega}{2} \hat{\sigma}_+ e^{-i\omega_L t}$$

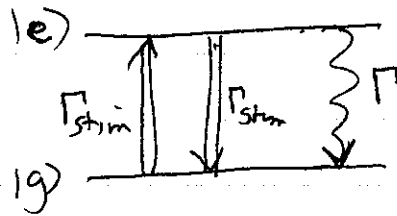
$$\mathcal{D}(\omega_L) = \text{atomic lineshape (normalized)} = \frac{\Gamma/2\pi}{\Delta^2 + \Gamma^2/4}$$

$$\Rightarrow \Gamma_{\text{stim}} = \frac{2\pi}{\hbar^2} \left(\frac{\hbar^2 \Omega^2}{4} \right) \frac{\Gamma/2\pi}{\Delta^2 + \Gamma^2/4} = \frac{\Omega^2 \Gamma}{4 \Delta^2 + \Gamma^2/4} \quad \checkmark$$

⇒ Rate equations

$$\dot{N}_e = -(\Gamma + \Gamma_{stim}) N_e + \Gamma_{stim} N_g$$

$$\dot{N}_g = -\Gamma_{stim} N_g + (\Gamma + \Gamma_{stim}) N_e$$



$$\dot{N}_e = -\dot{N}_g$$

$$N_e + N_g = 1$$

Steady state ⇒ detailed balance $\dot{N}_e = \dot{N}_g = 0$

⇒ Steady state populations

$$\dot{N}_e = 0 = -(\Gamma + \Gamma_{stim}) N_e^{s.s.} + \Gamma_{stim} (1 - N_e^{s.s.})$$

$$\Rightarrow N_e^{s.s.} = \frac{\Gamma_{stim}}{\Gamma + 2\Gamma_{stim}} = \frac{S/2}{1+S}$$

$$N_g^{s.s.} = \frac{1+S/2}{1+S}$$

Here I have defined:

Saturation parameter: $S \equiv \frac{2\Gamma_{stim}}{\Gamma} = \frac{\Omega^2/2}{\Delta^2 + \frac{\Gamma^2}{4}}$

For a given detuning and "oscillator strength", the saturation parameter determines the intensity ($\sim \Omega^2$) at which we pump substantial population into the excited state. When $S \ll 1$

$$N_g \approx 1 - S/2, \quad N_e \approx S/2 \Rightarrow \text{Linear regime}$$

when $s \rightarrow \infty$, $N_e^{s.s.} \approx N_g \approx \frac{1}{2}$

\Rightarrow the transition is "saturated".

Resonant behavior: $\Delta = 0$

$$S_0 = \frac{2\Omega^2}{\Gamma^2} \Rightarrow S = 1 \text{ when } \Omega = \frac{\Gamma}{\sqrt{2}}$$

Since $\Omega = \frac{\text{deg} E}{\hbar} \propto E \Rightarrow \Omega^2 \propto E^2 \propto I$ (intensity)

$$\Rightarrow S_0 = \frac{I}{I_{\text{sat}}} \text{ where } I_{\text{sat}} \equiv \text{saturation intensity}$$

$$I_{\text{sat}} = \frac{I}{S_0} = \frac{I}{\frac{2\Omega^2}{\Gamma^2}} = \frac{\hbar^2 \Gamma^2}{2 \text{deg}^2} \frac{I}{E^2} = \left(\frac{\hbar^2 \Gamma^2}{2 \text{deg}^2} \right) \left(\frac{c}{8\pi} \frac{E^2}{E^2} \right)$$

$$I_{\text{sat}} = \frac{c}{6\pi} \frac{\hbar^2 \Gamma^2}{\text{deg}^2} \quad \text{c.g.s units}$$

$$\left[\text{Aside: We will see that } \Gamma = \frac{4}{3} \left(\frac{\omega_{eg}}{c} \right)^3 \frac{\text{deg}^2}{\hbar} \Rightarrow \text{deg}^2 = \frac{3}{4} \hbar \Gamma \left(\frac{c}{\omega_{eg}} \right)^3 \right]$$

$$\Rightarrow I_{\text{sat}} = (\hbar \omega_{eg}) \left(\frac{1}{6\pi \left(\frac{c}{\omega_{eg}} \right)^2} \right) \left(\frac{\Gamma}{2} \right) = \frac{\text{Energy}}{\text{time} \cdot \text{Area}}$$

Here I have used $\sigma_{\text{abs}} = 6\pi \left(\frac{c}{\omega_{eg}} \right)^2 =$ absorption cross-section

$$I_{\text{sat}} = \frac{\hbar \omega_{eg}}{\sigma_{\text{abs}}} \frac{\Gamma}{2}$$

for 2-level atom

Off-resonance

$$S = \frac{S_0}{1 + \frac{4\Delta^2}{\Gamma^2}} = \frac{I/I_{\text{sat}}}{1 + \frac{4\Delta^2}{\Gamma^2}}$$

Saturation falls off like $\frac{\Gamma^2}{\Delta^2}$ for $\Delta \gg \Gamma$

Absorption cross-section and scattering

Recall definition of cross-section:

Given incident intensity I , the absorbed power is

$$P_{\text{abs}} = I \sigma_{\text{abs}}$$

$$\neq = (\text{Rate of absorption}) \times \hbar\omega$$

$$\text{Rate of absorption} = \Gamma_{\text{stim}} = \frac{s\Gamma}{2}$$

$$\Rightarrow \frac{s\Gamma}{2} \hbar\omega = I \sigma_{\text{abs}} = \left(\frac{I/I_{\text{sat}}}{1 + \frac{4\Delta^2}{\Gamma^2}} \right) \frac{\Gamma}{2} \hbar\omega$$

$$\Rightarrow \text{On resonance} \quad I \sigma_{\text{abs}} = \left(\frac{I}{I_{\text{sat}}} \right) \frac{\Gamma}{2} \hbar\omega_{\text{eg}}$$

$$\Rightarrow \boxed{I_{\text{sat}} = \frac{\hbar\omega_{\text{eg}}}{\sigma_{\text{abs}}} \frac{\Gamma}{2}} \quad \text{As before}$$

For two level atom rate of absorption \rightarrow Photon is Scattered

$$\Rightarrow \text{Scattering rate} \quad \boxed{\frac{s\Gamma}{2} = \frac{I}{\hbar\omega} \sigma_{\text{abs}}}$$

Saturation and Power Broadening

Let us return to the steady-state coherence and polarizability. The lineshape we found was for weak excitation, i.e., $S \ll 1$

More generally

$$\rho_{ge}^{s.s.} = \left(\frac{\Omega/2}{\Delta - i\Gamma/2} \right) (\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.})$$

\Rightarrow Steady state Bloch vector components

$$W^{s.s.} = \rho_{ee}^{s.s.} - \rho_{gg}^{s.s.} = \boxed{\frac{-1}{1+S}}$$

$$U^{s.s.} = 2 \operatorname{Re}(\rho_{ge}^{s.s.}) = \left(\frac{\Omega \Delta}{\Delta^2 + \Gamma^2/4} \right) \left(\frac{1}{1+S} \right)$$

$$\Rightarrow \boxed{U^{s.s.} = \frac{-2\Delta}{\Omega} \frac{S}{1+S}}$$

$$V^{s.s.} = 2 \operatorname{Im}(\rho_{ge}^{s.s.}) = \left(\frac{\Omega \Gamma/2}{\Delta^2 + \Gamma^2/4} \right) \left(\frac{1}{1+S} \right)$$

$$\Rightarrow \boxed{V^{s.s.} = \frac{-\Gamma}{\Omega} \frac{S}{1+S}}$$

$$\text{For } S \ll 1 \quad \begin{cases} U^{s.s.} \approx -\frac{2\Delta}{\Omega} S = \frac{\Delta\Omega}{\Delta^2 + \frac{\Gamma^2}{4}} \\ V^{s.s.} \approx \frac{\Gamma}{\Omega} S = \frac{\Gamma\Omega/2}{\Delta^2 + \frac{\Gamma^2}{4}} \end{cases}$$

For $s \gg 1$, $\omega \rightarrow 0$ Saturation

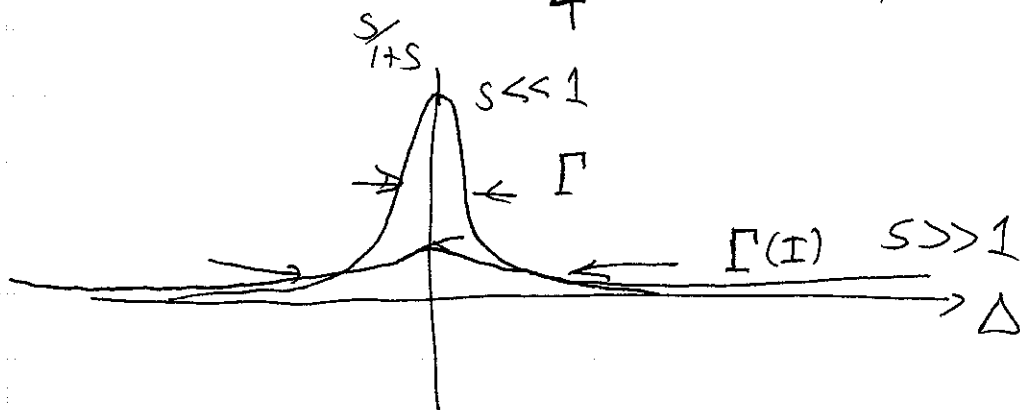
$$\Omega \gg \Delta, \Gamma \Rightarrow u, v \rightarrow 0$$

\Rightarrow Saturation = Completely mixed state

More details:

$$\frac{s}{1+s} = \frac{I/I_{\text{sat}}}{1 + \frac{4\Delta^2}{\Gamma^2} + I/I_{\text{sat}}} = \frac{\frac{2\Omega^2}{\Gamma^2}}{1 + \frac{4\Delta^2}{\Gamma^2} + \frac{2\Omega^2}{\Gamma^2}}$$

$$\Rightarrow \frac{s}{1+s} = \frac{\Omega^2/2}{\Delta^2 + \frac{\Gamma^2}{4} (1 + I/I_{\text{sat}})}$$



"Power Broadened line width"

$$\Gamma(I) = \Gamma \sqrt{1 + I/I_{\text{sat}}}$$