We have seen that an equivalence class of "unravellings" of the master equation arises from a "unitary remixing" of the Kraus operators \( \mathcal{E}_n \hat{\rho} = \int dt \hat{E}_n \hat{L}_n (\hat{a}) \hat{a} \; ; \; n = 1, 2, \ldots, m. \) This is not the full story. A more general equivalence class includes \( \hat{M}_o = 1 - \frac{i}{\hbar} \hat{H}_o dt \).

Let us focus attention on the example of the damped Simple Harmonic Oscillator, e.g. a single mode of an optical cavity with partially transparent mirror.

\[
\begin{bmatrix}
\hat{a} \\
\hat{a}^+ \\
\end{bmatrix}
\]

Master equation: \( \frac{d \hat{\rho}}{dt} = \left[ \frac{i}{\hbar} \hat{H} , \hat{\rho} \right] - \frac{1}{2} \left( \hat{a} \hat{a}^\dagger \hat{\rho} + \hat{\rho} \hat{a}^\dagger \hat{a} - 2 \hat{a}^\dagger \hat{a} \hat{\rho} \right) \)

\[ \Rightarrow \hat{\rho}(t+dt) = \hat{M}_o \hat{\rho}(t) \hat{M}_o^+ + M_1 \hat{\rho}(t) M_1^+ \\
\]
where \( \hat{M}_o = 1 - \frac{i}{\hbar} \hat{H}_o dt - \frac{\hat{a}^\dagger \hat{a}}{2} \)

\( \hat{M}_1 = \int \! dt \hat{a} \)

Consider the the remixing to order \( O(dt^2) \)

\[
\begin{bmatrix}
\hat{K}_o \\
\hat{K}_1 \\
\end{bmatrix} = \begin{bmatrix}
1 - \frac{\hat{H}_o}{2} \hat{a}^\dagger \hat{a} & -\int \! dt \hat{a} \hat{a}^\dagger \\
\int \! dt \hat{a} \hat{a}^\dagger & 1 - \frac{\hat{H}_o}{2} \hat{a}^\dagger \hat{a}
\end{bmatrix} \begin{bmatrix}
\hat{M}_o \\
\hat{M}_1 \\
\end{bmatrix}
\]
\[ \hat{K}_0 = 1 - i \frac{dt}{\hbar} \hat{A} - \frac{\hbar}{2} \frac{d}{dt} \hat{a}^+ \hat{a} - i \hbar \hat{a}^+ \hat{a} - \frac{\hbar}{2} \frac{d}{dt} |\psi|^2 \]

\[ = 1 - i \frac{dt}{\hbar} \left[ \hat{A} + \left( \alpha^* \hat{a} - \alpha \hat{a}^+ \right) \right] - \frac{\hbar}{2} \frac{d}{dt} \left( \hat{a}^+ \hat{a} \right) \]

\[ \hat{K}_1 = i dt \left( \hat{a} + \alpha \right) \]

You can easily prove to yourself \( \hat{P}^{(t+dt)} = \sum_{\mu=0}^{4} \frac{\hat{K}_{\mu}}{\hbar} \hat{P}^{(t)} \hat{K}_{\mu}^+ \)

This unitary \( \hat{K} \) operation remixed the system. This transforms

\[ \hat{a} \Rightarrow \hat{a} + \alpha \]

\[ \hat{A} \Rightarrow \hat{A} + \frac{\alpha^* \hat{a} - \alpha \hat{a}^+}{2i} \]

Physically, the \( \hat{K} \) operation corresponds to the effect on the system when a measurement is made on the environment. Since the new Lindblad operator is \( \sqrt{\hbar dt} \left( \hat{a} + \alpha \right) \), we recognize this as a homodyne measurement of the field that leaked out of the cavity.

\[ \text{highly transparent beam splitter} \]

\[ \text{Detector} \]

\[ \text{local oscillator} \]

(We define \( \alpha = r \beta \), where \( r \to 0 \)).
Mixing the cavity mode with a local oscillator we detect the beat note between them. Given the assumed macroscopic size of $\alpha$, the signal seen by the detector is more naturally described as a continuous (noisy) photocurrent rather than discrete set of photocounts.

\[ \text{Signal} \]

\[ \text{direct photocounting} \quad \text{homodyne photocurrent} \]

Of course, the homodyne signal itself is composed of discrete photocounts, but each discrete event has negligible information. To see this explicitly, consider the change in the state upon photodetection in homodyne

\[ |\psi\rangle - \frac{\hat{a} |\psi\rangle}{\langle \psi | \hat{a}^\dagger \psi \rangle} = |\psi\rangle - \left(1 + \frac{\hat{a}^\dagger}{\langle \psi | \hat{a}^\dagger \psi \rangle} \right) |\psi\rangle \]

\[ \Delta \psi = \frac{\hat{a}}{\langle \psi | \hat{a}^\dagger \psi \rangle} \Rightarrow \langle \Delta \psi | \Delta \psi \rangle = \frac{|\langle \psi | \hat{a}^\dagger \hat{a} \psi \rangle|^2}{\langle \psi | \hat{a}^\dagger \psi \rangle^2} \]

Thus when $|\langle \psi | \hat{a}^\dagger \hat{a} \psi \rangle|^2 \gg |\langle \psi | \hat{a}^\dagger \psi \rangle|^2$, the back action on a individual click is negligible. This motivates us to consider a quantum trajectory coarse-grained over many clicks so that the measurement record is continuous rather than discrete.
From Discrete Jump Processes to Continuous Wiener Processes

To derive a stochastic differential equation in the case of the homodyne detection, we coarse grain over a time interval $\Delta t$ such that

$$\frac{1}{\gamma_1^2} \ll \Delta t \ll \frac{1}{\gamma_1^2}$$

As shown above, if $\frac{\langle \alpha \alpha \rangle}{\langle \gamma_1^2 \rangle} \approx e \ll 1$, there will be many counts, but negligible back action.

- We scale $\Delta t$ such that $\Gamma \Delta t = \mathcal{O}(e^{\frac{1}{2}})$

$$\Rightarrow \left\{ \begin{array}{l}
\Gamma \Delta t |x|^2 = \mathcal{O}(e^{\frac{1}{2}}) \\
\Gamma \Delta t |x| = \mathcal{O}(e^{-\frac{1}{2}})
\end{array} \right.$$  

Using these, we consider a stochastic differential equations based on the Kraus operators $\hat{K}_0^2, \hat{K}_1^2, \hat{K}_2^2$

Unnormalized $\hat{\Psi}(t+\Delta t) = dN_0 \hat{K}_0^2 \hat{\Psi}(t) + dN_1 \hat{K}_1^2 \hat{\Psi}(t) \frac{1}{\sqrt{1-\Pi_1}} \frac{1}{\sqrt{1-\Pi_1}}$

Note that we have partially renormalized the post-measurement state according to $\Pi_1 = \Gamma \Delta t + |x|^2$

This ensures that $\hat{\Psi}(t+\Delta t) - \hat{\Psi}(t)$ is of $\mathcal{O}(e)$

- $\hat{K}_1^2 \hat{\Psi}(t) = \left( \frac{\alpha}{\sqrt{\Pi_1}} - 1 \right) \hat{\Psi}(t)$ (note: I have set the phase of local oscillator to $2\pi\omega$)

- $\hat{K}_2^2 \hat{\Psi}(t) = \left( 1 - \frac{\hat{x}^2 \hat{a} + \frac{\Gamma}{2} \Delta t \hat{a}^2 - \Gamma \Delta t \hat{x}^2 \hat{a}}{\sqrt{1-\Pi_1}} \right) \hat{\Psi}(t)$
\( \Phi \) with \( dN_0 = 1 - dN_1 \) and \( dN_1 dt = 0 \)

\[
\Rightarrow \frac{d\tilde{\Psi}(t)}{dt} = dt \left( \frac{-i\hat{A}}{\hbar} - \frac{\Gamma}{2} a^+ a - \Gamma a^+ a^* a \right) \tilde{\Psi}(t) + dN_1 \frac{a}{\hbar} \tilde{\Psi}(t)
\]

Now coarse-grain \( \int d\tilde{\Psi}(t) = \Delta \tilde{\Psi}(t) \)

\[
\Rightarrow \Delta \tilde{\Psi}(t) = \Delta t \left( \frac{-i\hat{A}}{\hbar} - \frac{\Gamma}{2} a^+ a - \Gamma a^+ a^* a \right) \tilde{\Psi}(t) \\
+ \frac{\Delta N(t)}{\hbar} \frac{a}{\hbar} \tilde{\Psi}(t)
\]

Having assumed \( \int \tilde{\Psi} \right) dt = \tilde{\Psi} \right) \Delta t \) (i.e. \( \tilde{\Psi} \right) unchanged in interval \( \Delta t \))

The stochastic variable \( \int dt = \Delta N(t) \) represents the accumulated number of counts in interval \( \Delta t \).

In the macroscopic limit, according to the central limit theorem, the Poisson distribution will be well-approximated by a Gaussian distribution.

\[
\Delta N(t) = \overline{\Delta N(t)} + \sqrt{\Delta N(t)} \Delta W(t)
\]

where \( \Delta W(t) \) is a Gaussian random variable (Wiener process) with zero mean and variance \( \langle (\Delta W(t))^2 \rangle = \Delta t \)
and \( \Delta N(t) = \int \langle \Psi(t) | K_{2t}(0) \hat{L}_{1}(0) ^t | \Psi(t) \rangle \).

\[
\Delta N(t) = \Gamma \Delta t \left( \langle a \rangle^2 + 2\langle \hat{a} \rangle \langle \hat{a} \rangle \right) = \Gamma \Delta t \left( \langle \hat{a}^2 \rangle + \langle \hat{a} \rangle^2 \right) \approx \Gamma \Delta t \langle \hat{a} \rangle^2 \text{ for } \langle \hat{a} \rangle \gg 1
\]

\[
\hat{X} = \frac{\hat{a} + \hat{a}^*}{\sqrt{2}} \Rightarrow \hat{X} = \frac{\hat{a} + \hat{a}^*}{\sqrt{2}} \phi = 0
\]

To understand this expression, consider the homodyne current as a stochastic variable

\[
I_{\text{hom}}(t) = \overline{I}_{\text{hom}}(t) + \int \overline{I}_{\text{hom}} \xi(t)
\]

where \( \xi(t) \) is Gaussian "white noise" representing the shot-noise in the detector with

\[
\overline{I}_{\text{hom}} = \langle \Psi | K_{1t} K_{2t} | \Psi \rangle \approx \Gamma \left( \langle \hat{X} \rangle^2 + 2 \langle \hat{X} \rangle \langle \hat{X} \rangle \right)
\]

\[
\langle \xi(t') \xi(t) \rangle = \delta(t-t') \text{ ensemble average}
\]

\[
\Delta W(t) = \int_{0}^{t} \xi(t') dt' \text{ is a continuous random variable}
\]

\[
\langle \Delta W^2(t) \rangle = \Delta t
\]

\[
\Delta N(t) = \Gamma \Delta t \left( \langle \hat{x}^2 \rangle + \langle \hat{x} \rangle^2 \right) + \int \Gamma \langle |X| \Delta W \rangle
\]

(Note: I have dropped the term \( \langle \sqrt{\langle \hat{x}^2 \rangle \hat{x}} \rangle \to \delta \langle \hat{x} \rangle \) to \( \delta \langle \hat{x} \rangle \))
We now take the limit treating $\Delta t \to dt$, a differential for $|x|$ large enough, differential

$$\Delta N(t) \Rightarrow \Gamma dt (1|\Delta x|^2 + 2|\Delta x| \langle \Delta \hat{X} \rangle) + \sqrt{\Gamma} \Delta x \Delta W(t)$$

The quantity $\Delta W(t)$ is known as the "Wiener interval" in stochastic calculus. For a more rigorous discussion, see "Handbook of Stochastic Methods", C.W. Gardiner, Chapter 4.

The (unnormalized) form of the stochastic Schrödinger equation can then be written

$$|\Psi(t)\rangle = \left[ i\hbar \left\{ \frac{-i\hbar^2}{2m \Delta^2} - \frac{\Delta}{\hbar} \left[ \hat{X} \Delta t + \sqrt{\Delta t} \Delta W(t) \right] \right\} \right] |\Psi(t)\rangle$$

In a less rigorous (but more physically transparent) form

$$\frac{d}{dt} |\Psi(t)\rangle = \frac{-i\hbar}{\hbar^2} \left\{ \frac{-i\hbar^2}{2m \Delta^2} - \frac{\Delta}{\hbar} \left[ \hat{X} \Delta t + \sqrt{\Delta t} \Delta W(t) \right] \right\} |\Psi(t)\rangle$$

where $\Delta W(t)$ is the stochastic homodyne current (minus the constant component).

This is the form originally derived by Carmichael.
In this form, the quantum state evolve continuously but randomly. It "diffuses" in Hilbert space according to Gaussian noise $dW$ arising solely from the shot noise from local oscillator rather than from the quantum uncertainty in the state of the cavity mode.

"Quantum State Diffusion" was first studied by Gisin and GR (PRL 52 1654 (1984)) as a way of showing how "collapse" of the wave function could be seen as a dynamical process. In the form written by Gisin and GR (equivalent)

$$d|\psi(t)\rangle = \left( -\frac{i}{\hbar} \text{Heff} dt + \left( dt \langle \hat{a}^\dagger \hat{a} \rangle + \sqrt{r} dW \right) \hat{a} \right) |\psi(t)\rangle$$

we recognize this as a quantum trajectory in which the environment is monitored via heterodyne rather than homodyne detection

(See Wiseman & Milburn, PRD 47 1652 (1993))

The connection with the work of Gisin and others shows the connection between this type of unraveling of the master equation and the theory of continuous measurement, whereby one continuously learns about an observable (more below).
Recovering the Master Equation

Let us show that this unravelling leads to the master equation for the damped S+O when averaged over realizations: We have, in unnormalized form,

\[ \hat{\Psi}(t+\Delta t) = \left[ (1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt) + (2\Gamma \langle \hat{X} \rangle dt + \sqrt{\Gamma} dW) \right] \hat{\Psi}(t) \]

so

\[ \hat{\rho}(t+\Delta t) = \frac{1}{\text{Tr}(\hat{\rho}(t))} \hat{\Psi}(t+\Delta t) \langle \hat{\Psi}(t+\Delta t) \rangle \]

\[ = \hat{\rho}(t) - \frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}(t)] dt + \Gamma dW^2 \hat{\rho} \hat{\rho}^+ + (2\Gamma \langle \hat{X} \rangle dt + \sqrt{\Gamma} dW) (\hat{\rho} \hat{\rho}^+ + \hat{\rho}^+ \hat{\rho}) \]

To get the normalized form, divide by

\[ \text{Tr}(\hat{\rho}(t+\Delta t)) = \frac{1}{\text{Tr}(\hat{\rho}(t))} \]

\[ \text{Tr}(\hat{\rho}) \cdot -\frac{i}{\hbar} \text{Tr}(\hat{H}_{\text{eff}} \hat{\rho} - \hat{\rho} \hat{H}_{\text{eff}}^+) + \Gamma dt \text{Tr}(\hat{\rho} \hat{a}^+ \hat{a}) \]

\[ + (2\Gamma \langle \hat{X} \rangle dt + \sqrt{\Gamma} dW) \text{Tr}(\hat{\rho} \hat{a}^+ \hat{a}) \]

\[ = \text{Tr}(\hat{\rho}) \left( 1 + (2\Gamma \langle \hat{X} \rangle dt + \sqrt{\Gamma} dW) \frac{\langle \hat{a}^+ \hat{a} \rangle}{\langle \hat{a}^+ \hat{a} \rangle} \right) \]

Here I have used \( dW^2 = dt \)

and \[ \text{Tr}((\hat{H}_{\text{eff}} - \hat{H}_{\text{eff}})^\dagger \hat{\rho}) = -i\hbar \Gamma \langle \hat{a}^+ \hat{a} \rangle \]
\[
\text{Tr}[\hat{\rho}(t+\Delta t)] = \frac{1}{\text{Tr}(\hat{\rho})} \left( 1 - (2\Gamma \langle \hat{\mathbf{x}} \rangle \Delta t + \sqrt{\Gamma} \Delta \mathcal{W}) \langle \hat{\mathbf{x}} \rangle \right)
\]

\[
+ \left( \sqrt{\Gamma} \Delta \mathcal{W} \langle \hat{\mathbf{x}} \rangle \right)^2
\]

Again, having used the rules of stochastic calculus, \( \Delta t \Delta \mathcal{W} = 0 \); \( \Delta \mathcal{W}^2 = \Delta t \).

Normalizing, the conditional master equation:

\[
\hat{\rho}_c(t+\Delta t) = \frac{\hat{\rho}_c(t+\Delta t)}{\text{Tr}(\hat{\rho}(t+\Delta t))}
\]

\[
= \hat{\rho}_c(t) - \frac{1}{\hbar} \left[ \hat{H}_{\text{eff}}, \hat{\rho}_c(t) \right] \Delta t + \Gamma d + \hat{a} \hat{\rho}_c \hat{a}^+ \left( 1 - \sqrt{\Gamma} \langle \hat{\mathbf{x}} \rangle \Delta \mathcal{W} \right)
\]

\[
+ (2\Gamma \langle \hat{\mathbf{x}} \rangle \Delta t + \sqrt{\Gamma} \Delta \mathcal{W} \langle \hat{\mathbf{x}} \rangle \left( \Delta \mathcal{W} \langle \hat{\mathbf{x}} \rangle \right)) \left( \hat{a} \hat{\rho}_c + \hat{\rho}_c \hat{a}^+ \right)
\]
(Next Page)

Since the system goes to another, considered as the measurement
of the environment, so the true pure state of the
observation which is made a perfect measurement.

Because we derived it from a stochastic Schrödinger
equation, there is a position ambiguity. This is

\[ \frac{\mathbf{H} + \mathbf{p}^2}{2} + (\mathbf{p} \cdot \mathbf{d} + \mathbf{a} \cdot \mathbf{d}) + 1 = \]

\[ \frac{\mathbf{H} + \mathbf{p}^2}{2} + \mathbf{p} \cdot \mathbf{d} + \mathbf{a} \cdot \mathbf{d} = \frac{\partial}{\partial t} \]

\[ \frac{\partial}{\partial t} = -\frac{i}{\hbar} \left[ \mathbf{H}, \mathbf{p} \right] \]

expanded master equation.

The uncontrolled density operator drive do

The essential answer over importance, \( \Delta W = 0 \)

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The close equation is known as the

\[ \left( \frac{\partial}{\partial t} \mathbf{H} + \mathbf{p} \cdot \mathbf{d} + \mathbf{a} \cdot \mathbf{d} \right) \]

\[ \frac{\partial}{\partial t} = \frac{i}{\hbar} \left[ \mathbf{H}, \mathbf{p} \right] \]
We can use our results to write a stochastic master Eqn not just as an imagined "unravelling" of unconditioned master Eqn, but also as the dynamics of a conditioned state given a real measurement. Such measurements are typically perfect. We can include, e.g., an imperfect detection efficiency, simply in the SME

\[ \frac{d\rho_c}{dt} = -i \frac{\hbar}{\eta} [H_{ck}, \rho_c] dt + \Gamma dt \rho_c \rho_c^+ \\
+ \sqrt{\eta} dW (\rho_c + \rho_c^+ - \langle \rho_c \rangle) \]

where \( \eta \) is the detection efficiency \( \eta \leq 1 \).

The SME in this form was first written by V.P. Belaknin in the context of the study of "continuous quantum measurement", i.e., the continuous observation of a quantum system by a probe. Clearly, this unravelling of the master Eqn via homodyne measurement is intimately connected with the theory of "quantum filtering" and "continuous measurement", which we take up in the next lecture.