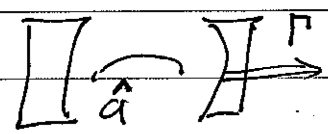
Lect 29   Continuous Stochastic Schrödinger Eqn

We have seen that an equivalence class of "unravellings" of the master equation arises from a "unitary remixing" of the Kraus operators $\{\hat{M}_\mu = \sqrt{dt}\,\hat{L}_\mu\}$; $\mu = 1, 2, \ldots m$. This is not the full story. A more general equivalence class includes $\hat{M}_0 = \mathbb{1} - \frac{i}{\hbar}\hat{H}_{eff}\,dt$.

Let us focus attention on the example of the damped Simple Harmonic Oscillator; e.g. a single mode of of optical cavity with partially transparent mirror:



Master equation: $\dfrac{d\hat{\rho}}{dt} = \left[\frac{i}{\hbar}\hat{H}, \hat{\rho}\right] - \frac{\Gamma}{2}\left(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right)$

$$\Rightarrow \quad \hat{\rho}(t+dt) = \hat{M}_0\,\hat{\rho}(t)\,\hat{M}_0^\dagger + \hat{M}_1\,\hat{\rho}(t)\,\hat{M}_1^\dagger$$

where
$$\hat{M}_0 = \mathbb{1} - \frac{i}{\hbar}\hat{H}\,dt - \frac{\Gamma dt}{2}\hat{a}^\dagger\hat{a}$$
$$\hat{M}_1 = \sqrt{\Gamma dt}\,\hat{a}$$

Consider the the remixing to order $O(dt)$

$$\begin{bmatrix} \hat{K}_0 \\ \hat{K}_1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{\Gamma dt}{2}|\alpha|^2 & -\sqrt{\Gamma dt}\,\alpha^* \\ \sqrt{\Gamma dt}\,\alpha & 1 - \frac{\Gamma dt}{2}|\alpha|^2 \end{bmatrix} \begin{bmatrix} \hat{M}_0 \\ \hat{M}_1 \end{bmatrix}$$

$$\Rightarrow \hat{K}_0 = 1 - \frac{i\,dt}{\hbar}\hat{H} - \frac{\Gamma}{2}dt\,\hat{a}^\dagger\hat{a} - \Gamma dt\,\alpha^*\hat{a} - \frac{\Gamma}{2}dt\,|\alpha|^2$$

$$= 1 - \frac{i\,dt}{\hbar}\left[\hat{H} + \left(\frac{\alpha^*\hat{a} - \alpha\hat{a}^\dagger}{2i}\right)\right] - \frac{\Gamma}{2}dt\,(\hat{a}+\alpha)^\dagger(\hat{a}+\alpha)$$

$$\hat{K}_1 = \sqrt{dt\,\Gamma}\,(\hat{a}+\alpha)$$

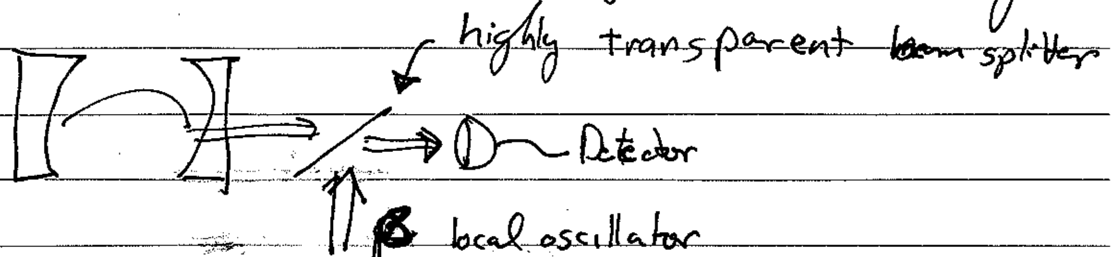
You can easily prove to yourself $\hat{\rho}(t+dt) = \sum_{\mu=0}^{1} \hat{K}_\mu \hat{\rho}(t) \hat{K}_\mu^\dagger$

This unitary remixing thus transforms

$$\hat{a} \Rightarrow \hat{a} + \alpha$$

$$\hat{H} \Rightarrow \hat{H} + \frac{\alpha^*\hat{a} - \alpha\hat{a}^\dagger}{2i}$$

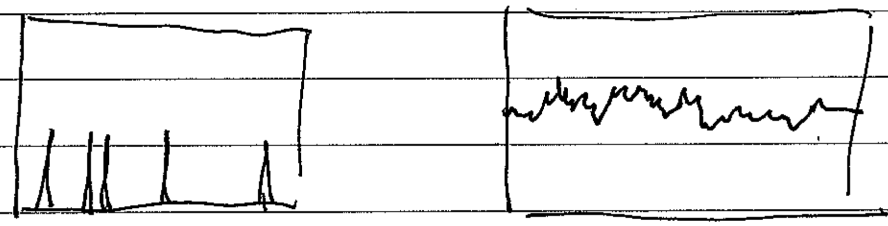
Physically, the Linblad operators corresponds to the effect on the system when a measurement is made on the enbironment. Since the new Linblad operator in $\sqrt{\Gamma dt}\,(\hat{a}+\alpha)$, we recognize this as <u>homodyne</u> measurement of the field that leaked out of the cavity



(We define $\alpha = r\beta$ when $r \to 0$, $\beta \to \infty$)

Mixing the cavity mode with a local oscillator we detect the beat note between them.
Given the assummed macroscopic size of α, the signal seen by the detector is more naturally described as a continuous (noisy) photocurrent rather than discrete set of photocounts

Signal



direct photocounting — homodyne photo-current

Of course, the homodyne signal itself is composed of discrete photocounts, but each discrete event has negligible information. To see this explicitly, consider the change in the state upon photo-detection in homodyne

$$|\psi\rangle - \frac{\hat{K}_1|\psi\rangle}{\|\hat{K}_1|\psi\rangle\|} = |\psi\rangle - \left(1 + \frac{\hat{a}}{|\alpha|}\right)|\psi\rangle$$

$$\Delta|\psi\rangle = \frac{\hat{a}}{|\alpha|}|\psi\rangle \Rightarrow \|\Delta|\psi\rangle\| = \frac{\sqrt{\langle\psi|a^\dagger a|\psi\rangle}}{|\alpha|}$$

Thus when $|\alpha|^2 \gg \langle\psi|a^\dagger a|\psi\rangle$, the back action on a individual click is negligible. This motivates us to consider a quantum trajectory coarse-grained over many clicks so that the measurement record is continuous rather than discrete.

From Discrete Jump Processes to Continuous Wiener Processes

To derive a stochastic differential equation in the case of the homodyne detection, we coarse grain over a time interval Δt

$$\frac{1}{\Gamma|\alpha|^2} << \Delta t << \frac{1}{\Gamma}$$

As shown above, if $\frac{\langle a^\dagger a\rangle}{|\alpha|^2} \equiv \epsilon << 1$, there will be many counts, but negligible back action.

- We scale Δt such that $\Gamma\Delta t = \mathcal{O}(\epsilon^{1/2})$

$$\Rightarrow \begin{cases} \Gamma\Delta t|\alpha|^2 = \mathcal{O}(\epsilon^{3/2}) \\ \Gamma\Delta t\,|\alpha| = \mathcal{O}(\epsilon^{-1/2}) \end{cases}$$

Using these, we consider a stochastic differential equations based on the Kraus operators $\{\hat{K}_0, \hat{K}_1\}$

Unnormalized $|\tilde{\Psi}(t+dt)\rangle = dN_0 \frac{\hat{K}_0|\Psi(t)\rangle}{\sqrt{1-\Pi_1}} + dN_1 \frac{\hat{K}_1|\Psi(t)\rangle}{\sqrt{\Pi_1}}$

Note that we have partially renormalized the post-measurement state according to $\Pi_1 \equiv \Gamma dt|\alpha|^2$.

This ensures that $|\tilde{\Psi}(t+dt)\rangle - |\Psi\rangle$ is of $\mathcal{O}(\epsilon)$

- $\frac{\hat{K}_1|\Psi(t)\rangle}{\sqrt{\Pi_1}} = \left(\frac{\hat{a}}{|\alpha|} - 1\right)|\Psi\rangle$ (note: I have set the phase of local oscillator to zero)

- $\frac{\hat{K}_0|\Psi(t)\rangle}{\sqrt{1-\Pi_1}} = \left(1 - \frac{i dt}{\hbar}\hat{H} - \frac{\Gamma dt}{2}\hat{a}^\dagger\hat{a} - \Gamma dt\,\alpha^*\hat{a}\right)$

with $dN_0 = 1 - dN_1$ and $dN_1 dt = 0$

$$\Rightarrow d|\tilde{\Psi}(t)\rangle = dt\left(\frac{-i}{\hbar}\hat{H} - \frac{\Gamma}{2}\hat{a}^\dagger\hat{a} - \Gamma\alpha^*\hat{a}\right)|\tilde{\Psi}\rangle + \frac{dN_1}{|\alpha|}\hat{a}|\tilde{\Psi}\rangle$$

Now coarse-grain $\int_t^{t+\Delta t} d|\tilde{\Psi}(t)\rangle = \Delta|\tilde{\Psi}(t)\rangle$

$$\Rightarrow \Delta|\tilde{\Psi}(t)\rangle = \Delta t\left(\frac{-i}{\hbar}\hat{H} - \frac{\Gamma}{2}a^\dagger a - \Gamma\alpha^*\hat{a}\right)|\tilde{\Psi}\rangle + \frac{\Delta N(t)}{|\alpha|}\hat{a}|\tilde{\Psi}\rangle$$

Having assumed $\int_t^{t+\Delta t}|\tilde{\Psi}\rangle dt = |\tilde{\Psi}\rangle\Delta t$

(i.e. $|\tilde{\Psi}\rangle$ unchanged in interval Δt)

The stochastic variable $\int_t^{t+\Delta t} dN(t) = \Delta N(t)$ represents the accumulated number of counts in interval Δt.
In the macroscopic limit, according to the central limit theorem, the Poisson distribution will be well-approximated by a Gaussian

$$\Delta N(t) = \overline{\Delta N(t)} + \sqrt{\frac{\overline{\Delta N(t)}}{\Delta t}}\,\Delta W(t)$$

where $\Delta W(t)$ is a Gaussian Random variable (Weiner process) with zero mean and variance $\langle \Delta W(t)^2\rangle = \Delta t$

and $\overline{\Delta N(t)} = \int_t^{t+\Delta t} \langle \Psi(t) | \hat{K}_1^\dagger(t) \hat{K}_1(t) | \Psi(t) \rangle$

$$= \Gamma \Delta t \langle (\hat{a}+\alpha)^\dagger (a+\alpha) \rangle$$

$$= \Gamma \Delta t \left(|\alpha|^2 + 2|\alpha| \langle \hat{X} \rangle + \cancel{\langle a^\dagger a \rangle} \right)$$ negligible to $\mathcal{O}(\epsilon)$

$$\hat{X} = \frac{\hat{a} e^{-i\phi} + \hat{a}^\dagger e^{i\phi}}{2} \rightarrow \frac{a + a^\dagger}{2}, \quad \phi = 0$$

To understand this expression, consider the homodyne current as a stochastic variable

$$I_{hom}(t) = \overline{I}_{hom}(t) + \sqrt{I_{hom}}\, \xi(t)$$

where $\xi(t)$ is Gaussian "white noise" representing the shot-noise in the detector with

$$\overline{I}_{hom} = \langle \Psi | \frac{K_1^\dagger K_1}{\Delta t} | \Psi \rangle = \Gamma (|\alpha|^2 + 2|\alpha| \langle \hat{X} \rangle)$$

$$\overline{\xi(t')\, \xi(t)} = \delta(t - t')$$ ensemble average

$\Rightarrow$ $\Delta W(t) = \int_t^{t+\Delta t} \xi(t')\, dt'$ is a continuous random variable

$$\langle \Delta W^2(t) \rangle = \Delta t$$

$\therefore$

$$\Delta N(t) = \Gamma \Delta t \left(|\alpha|^2 + 2|\alpha| \langle \hat{X} \rangle \right) + \sqrt{\Gamma} |\alpha| \Delta W$$

(Note: I have dropped the term $\sim \sqrt{|\alpha| \langle \hat{X} \rangle}$ to $\mathcal{O}(\epsilon)$

We now take the limit treating $\Delta t \to dt$, a ~~differential~~ differential for $|\alpha|$ large enough,

$$\Delta N(t) \Rightarrow \Gamma dt \left(|\alpha|^2 + 2|\alpha|\langle \hat{X} \rangle\right) + \sqrt{\Gamma}\,|\alpha|\, dW(t)$$

The quantity $dW(t)$ is know ~~the~~ the "Wiener interval" in Stochastic calculus. For a more rigorous discussion, see

{"Handbook of Stochastic Methods", C.W. Gardiner, Chapter 4}

The (unnormalized) form of the Stochastic Schrödinger equation can then be written

$$\boxed{d|\tilde{\psi}\rangle = \left[dt\left(\frac{-i}{\hbar}\hat{H} - \frac{\Gamma}{2}\hat{a}^\dagger\hat{a}\right) + \hat{a}\left(2\Gamma dt\,\langle \hat{X}\rangle + \sqrt{\Gamma}\,dW\right)\right]|\tilde{\psi}\rangle}$$

In a less rigorous (but more physically transparent) form

$$\frac{d}{dt}|\tilde{\psi}\rangle = \frac{-i}{\hbar}\underbrace{\hat{H}_{eff}}_{H - i\hbar\frac{\Gamma}{2}\hat{a}^\dagger\hat{a}}|\tilde{\psi}\rangle + \frac{\Delta I_{hom}(t)}{|\alpha|}\hat{a}|\tilde{\psi}\rangle$$

where $\Delta I_{hom}(t)$ is the stochastic homodyne current (minus the constant component).

This is the form originally derived by Carmichael.

Comments

- In this form, the quantum state evolve continuously but randomly. It "diffuses" in Hilbert space according to Gaussian noise dW arising solely from the shot noise from local oscillator rather than from the quantum uncertainty in the state of the cavity mode.

- "Quantum-State Diffussion" was first studied by Gisin ~~and~~ (PRL 52 1657 (1984)) as a way ~~of~~ showing how "collapse" of the wave function could be seen as a dynamical process. In the form written by Gisen and Percival (actually equivalent)

$$d|\tilde{\psi}\rangle = \left(-\frac{i}{\hbar} H_{eff}\, dt + \left(\Gamma dt \langle \hat{a}^\dagger \rangle + \sqrt{\Gamma}\, dW\right) \hat{a}\right) |\tilde{\psi}\rangle$$

we recognize this as a quantum trajectory in which the enivornment is monitored via heterodyne rather than homodyne detection

(See Wiseman & Milburn, PRA 47 1652 (1993))

- The connection with the work of Gisen and others shows the connection between this type of unravelling of the master equation and the theory of continuous measurement, whereby one continously learns about an observable (more below)

Recovering the Master Equation

Let us show that this unravelling leads to the master equation for the damped SHO when averaged over realizations. We have, in unnormalized form,

$$|\tilde{\Psi}(t+\Delta t)\rangle = \left[\left(1-\frac{i}{\hbar}\hat{H}_{eff}\,dt\right) + \left(2\Gamma\langle\hat{x}\rangle dt + \sqrt{\Gamma}\,dW\right)\hat{a}\right]|\tilde{\Psi}(t)\rangle$$

So
$$\tilde{\rho}(t+\Delta t) = |\tilde{\Psi}(t+\Delta t)\rangle\langle\tilde{\Psi}(t+\Delta t)|$$

$$= \tilde{\rho}(t) - \frac{i}{\hbar}[\hat{H}_{eff}, \tilde{\rho}(t)]\,dt + \Gamma\, dW^2\, \hat{a}\,\tilde{\rho}\,\hat{a}^\dagger$$

$$+ \left(2\Gamma\langle\hat{x}\rangle dt + \sqrt{\Gamma}\,dW\right)\left(\hat{a}\tilde{\rho} + \tilde{\rho}\hat{a}^\dagger\right)$$

To get the normalized form, divide by

$$\mathrm{Tr}(\tilde{\rho}(t+\Delta t)) =$$

$$\mathrm{Tr}(\tilde{\rho}) - \frac{i\,dt}{\hbar}\mathrm{Tr}\left(\hat{H}_{eff}\tilde{\rho} - \tilde{\rho}\hat{H}_{eff}^\dagger\right) + \Gamma\,dt\,\mathrm{Tr}(\tilde{\rho}a^\dagger a)$$

$$+ \left(2\Gamma\langle\hat{x}\rangle dt + \sqrt{\Gamma}\,dW\right)\mathrm{Tr}\left(\tilde{\rho}(a^\dagger + a)\right)$$

$$= \mathrm{Tr}(\tilde{\rho})\left(1 + \left(2\Gamma\langle\hat{x}\rangle dt + \sqrt{\Gamma}\,dW\right)\langle a + a^\dagger\rangle\right)$$

Here I have used $dW^2 = dt$

and $\mathrm{Tr}\left((H_{eff} - H_{eff}^\dagger)\rho\right) = -i\hbar\Gamma\langle\hat{a}^\dagger\hat{a}\rangle$

$$\therefore \quad \frac{1}{\mathrm{Tr}[\tilde{\rho}(t+\Delta t)]} = \frac{1}{\mathrm{Tr}(\tilde{\rho})} \; \frac{1}{1+(2\Gamma\langle\hat{x}\rangle dt + \sqrt{\Gamma}\, dW)2\langle\hat{x}\rangle}$$

$$= \frac{1}{\mathrm{Tr}(\tilde{\rho})}\Big(1-(2\Gamma\langle\hat{x}\rangle dt + \sqrt{\Gamma}\, dW)\, 2\langle\hat{x}\rangle + (\sqrt{\Gamma}\, dW\, 2\langle\hat{x}\rangle)^2 \Big)$$

$$= \frac{1}{\mathrm{Tr}(\tilde{\rho})}\left(1 - \sqrt{\Gamma}\, 2\langle\hat{x}\rangle\, dW\right) + \mathcal{O}(dt^2)$$

Again, having used the rules of stochastic calculus $dt\, dW = 0; \; dW^2 = dt$

Normalizing, the conditional master eqn.

$$\hat{\rho}_c(t+\Delta t) = \frac{\tilde{\rho}(t+\Delta t)}{\mathrm{Tr}(\tilde{\rho}(t+\Delta t))}$$

$$= \hat{\rho}_c(t) - \frac{i}{\hbar}\left[\hat{H}_{eff}, \hat{\rho}_c(t)\right] dt + \Gamma dt\, \hat{a}\, \hat{\rho}_c\, \hat{a}^\dagger$$

$$(1 - \sqrt{\Gamma}\, 2\langle\hat{x}\rangle\, dW)$$

$$+ (2\Gamma\langle\hat{x}\rangle dt + \sqrt{\Gamma}\, dW\, (1 - \sqrt{\Gamma}\, dW\, 2\langle\hat{x}\rangle))$$

$$(a\hat{\rho}_c + \hat{\rho}_c \hat{a}^\dagger)$$

$$\Rightarrow \boxed{d\rho_c = -\frac{i}{\hbar}[H_{eff}, \rho_c]\,dt + \Gamma\,dt\, a\rho_c a^\dagger + \sqrt{\Gamma}\,dW\left(a\rho_c + \rho_c a^\dagger - \langle a + a^\dagger\rangle \rho_c\right)}$$

The above equation is known as the "Stochastic master eqn" (SME). Since the ensemble average over trajectories, $\overline{dW} = 0$,

The unconditioned density operator obeys the expected master equation

$$\boxed{\begin{aligned} \frac{d\rho}{dt} = \frac{\overline{d\rho_c}}{dt} &= -\frac{i}{\hbar}[H_{eff}, \rho] + \Gamma a\rho a^\dagger \\ &= -\frac{i}{\hbar}[H, \rho] + \left[-\frac{\Gamma}{2}(a^\dagger a\rho + \rho a^\dagger a) + \Gamma a\rho a^\dagger\right] \end{aligned}}$$

The SME, as written, preserves purity. This is because we derived it from a stochastic Schrödinger equation which imagined a perfect measurement of the environment so that a pure state of the system goes to another, conditioned on the measurement outcome.

(Next Page)

We can use our results to write a Stochastic Master Eqn not just as an imagined "unravelling" of unconditioned master eqn, but also as the dynamics of a conditioned state given a real measurement. Such measurements are rarely perfect. We can include, e.g., an imperfect detection efficiency, simply in the SME

$$\boxed{d\rho_c = -\frac{i}{\hbar}[H_{eff}, \rho_c]\,dt + \Gamma dt\, a\rho_c a^\dagger + \sqrt{\Gamma\eta}\, dW\left(a\rho_c + \rho_c a^\dagger - \langle a + a^\dagger\rangle \rho_c\right)}$$

Where η is the detection efficiency $\eta \leq 1$

The SME in this form was first written by V.P. Belakin in the context of the study of "continuous quantum measurement", i.e., the continuous observation of a quantum system by a probe. Clearly, this unravelling of the master eqn via homodyne measurement is intimately connected with the theory of "quantum filtering" and "continuous measurement", which we take up in the next lecture.