Lecture 28: EPR and Bell's Inequality

• Hidden Variables?

Quantum mechanics is a statistical theory. That is, in general the results of experiments cannot be predicted with certainty; they are "random", and one thus assigns probabilities to possible outcomes. From where does this randomness arise? We are all familiar with the notion of uncertainty in our daily life. There are many things we cannot predict with unit probability, like the weather. However, we associate this lack of certainty with incomplete information. That is, if we knew with greater precision the initial conditions, and the dynamical model, we would be able to better predict the weather.

It is natural to assume that quantum mechanical uncertainty is of this sort. The randomness we observe in a quantum measurement would then be associated with incomplete knowledge of the underlying physics. Such a model of the microscopic world is known as a "hidden variable theory". The idea is that there are physical quantities that are somehow "hidden" from ~~us~~, but ~~effect~~ the observables we measure. If we were able to access these variables, we could predict, with certainty, the outcomes of experiments

- Einstein

Einstein is well known for his distaste for the "new" quantum theory. His most famous quote on the subject is his statement "God does not roll dice with the universe!" Whether Einstein's problem with Q.M. was that it involved randomness can be debated. What is clear is that he had great distaste for the "fuzzy" ~~and~~ almost "mystical" view of Q.M. established in the Copenhaggen school, and put forward by Niels Bohr. There is a famous set of debates on the subject between Bohr and Einstein which can be found in a nice compendium

"Quantum Theory and Measurement"
Edited by Wheeler and Zurek, Princeton Series in Physics, (Princeton University Press, 1983).

In particular Einstein was not happy with the idea of the "collapse of the wave function". I believe, this was partly because of the pervailing view at the time of the collapse being a "physical process". The measuring apparatus was thought to "force the quantum system into a state". ~~What~~ How could this happen. Did quantum systems not "have properties" until we measured them?

• EPR (Phys. Rev 47 777 (1935))

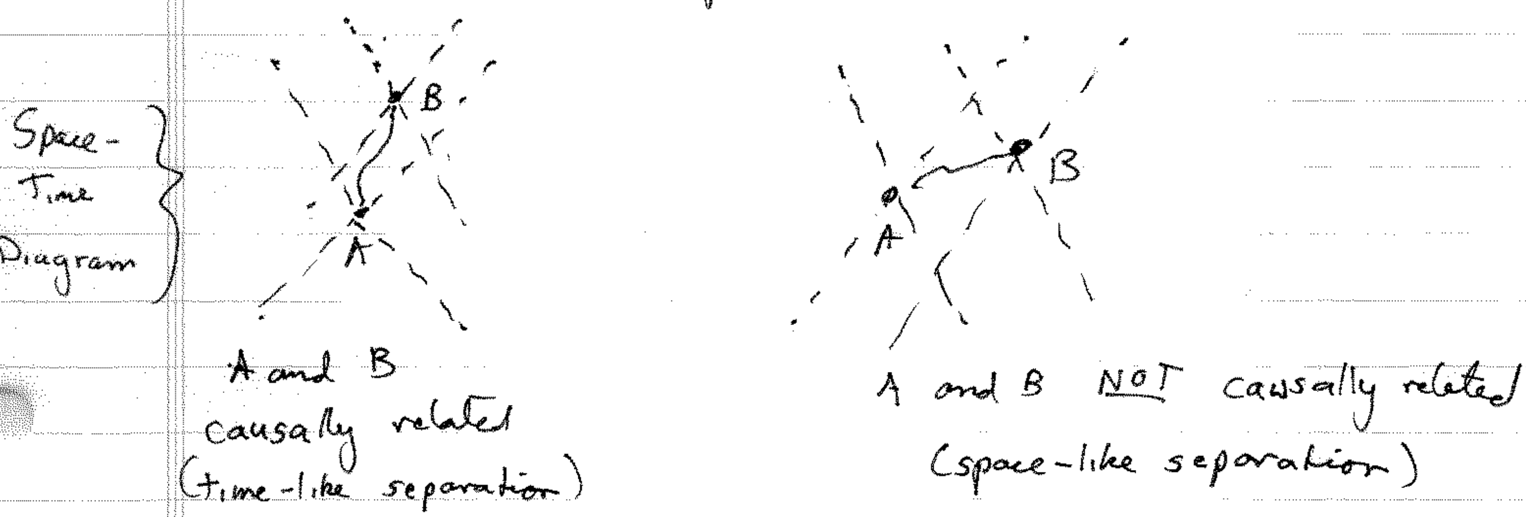
In 1935 Einstein, together with Podolsky and Rosen (known today as EPR) put a very fine point on the question of hidden variables. It ties in notions of causality with measurement and hidden variables.

The idea they wanted to get at is "objective properties". That is, measurements are uncovering properties of physical system that "exist", independent of the measurement. These objective properties they called "elements of physical reality" which they ~~describe~~ defined as:

"If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then the exists an element of physical reality corresponding to this quantity"

The argument put forward by EPR, which we will describe below, is that by their definition, we can can assign elements of reality to various observables, even ones that don't commute. Thus quantum mechanics is "incomplete", and would be complete by a hidden variable theory.

The heart of the EPR argument is based on "Einstein" causality, i.e. two events are causally related iff they lie within each others light cones. More simply stated, they are causally related if a signal can travel between them at a speed no greater than the speed of light.



EPR thus consider a <u>composite, bipartite, quantum system</u> that is <u>entangled</u>. Recall, the entangled (pure) states are ones that cannot be factorized

$$|\Psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\phi\rangle_B$$

Because of this, measurement statistics of A and B are <u>correlated</u>. However, these correlation are nothing like classical correlations as we will see.

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EPR considered a two-body system described by the wave function

$$\Psi(x_1, x_2) = \delta(x_1 - x_2 + x_0)$$

The momentum space wave function is

$$\tilde{\Psi}(p_1, p_2) = \int \frac{dx_1\, dx_2}{2\pi}\, \Psi(x_1, x_2)\, e^{-i(p_1 x_1 + p_2 x_2)}$$

$$= \int \frac{dx_1}{2\pi}\, e^{-i(p_1+p_2)x_1}\, e^{+ip_2 x_0}$$

$$\Rightarrow \tilde{\Psi}(p_1, p_2) = e^{ip_2 x_2}\, \delta(p_1 + p_2)$$

From these wave functions, we see that the state $|\Psi\rangle$ is simultaneously an eigenstate of $\hat{x}_2 - \hat{x}_1$ and $\hat{p}_1 + \hat{p}_2$ with respective eigenvalues x_0 and 0.

Note $[\hat{x}_1 - \hat{x}_2, \hat{p}_1 + \hat{p}_2] = 0$ so we can always find simultaneous eigenstates.

This fact implies that if A (known as "Alice") measure particle 1 and find value x_1 she knows the B ("Bob") will find particle 2 at $x_2 = x_1 + x_0$. If Alice measure momentum of particle 1 and find p_1, she knows Bob will find $p_2 = -p_1$.

Now, Alice and Bob can be separated at a space-like interval $\Rightarrow$ there is no causal relation between them.
Thus:

(1) Since Alice can (by measuring position of particle 1) predict with <u>certainty</u> the value of x_2 <u>without disturbing it</u> (since A + B are space-like separated), Alice would associate an element of reality with x_2

(2) Since Alice can (by measuring the momentum of particle 1) predict with <u>certainty</u> the value of p_2 <u>without disturbing it</u>, Alice would associate an element of reality with p_2

$\Rightarrow$ ~~Bot~~ Both x_2 and p_2 are objectively real with definitely predictable values

This contradicts the uncertainty principle.

$\Rightarrow$ <u>EPR says Q.M. is incomplete</u>

This kind of counterfactual reason is, of course, dangerous in Q.M. Alice does <u>either</u> a position measure <u>or</u> a momentum measurement, not both. However, EPR argue that because the value of either

x_2 ~~or~~ or p_2 can be predicted without disturbance, they must have been objective all along, even if we don't do the measurement.

This is a very compelling argument and caused great consternation. Bohr did reply, but in his usual "fuzzy" way that did not provide clear resolution. It wasn't until almost 30 years later that John S. Bell brought the issue from a philosophical argument to a distinction that has empirical (i.e. experimental) significance.

Before we get to Bell's analysis, let us first look at another version of the EPR system. Instead of considering measurements of position and momentum, let us consider measurements of spin. This idea is due to David Bohm Phy Rev 108 1070 (1957).

Bohm version of EPR

Consider a spin singlet for 2 spin-1/2 particles

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle_A \otimes |-_z\rangle_B - |-_z\rangle_A \otimes |+_z\rangle_B\right)$$

This state is entangled as in the original EPR example.

We recall, the singlet is an eigenstate of the total spin with eigenvalue $S=0$.

Thus $\left(\hat{\sigma}_n^{(A)} + \hat{\sigma}_n^{(B)}\right)|\Psi\rangle = 0$

for any axis $\vec{e}_n$ (recall $\hat{\sigma}_n \equiv \vec{e}_n \cdot \hat{\vec{\sigma}}_n$)

$$\Rightarrow \quad \hat{\sigma}_n^{(A)}|\Psi\rangle = -\hat{\sigma}_n^{(B)}|\Psi\rangle$$

In fact, $S=0$ is rotationally invariant

$$\Rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+_n\rangle_A \otimes |-_n\rangle_B - |-_n\rangle_A \otimes |+_n\rangle_B\right)$$

(up to an overall phase)

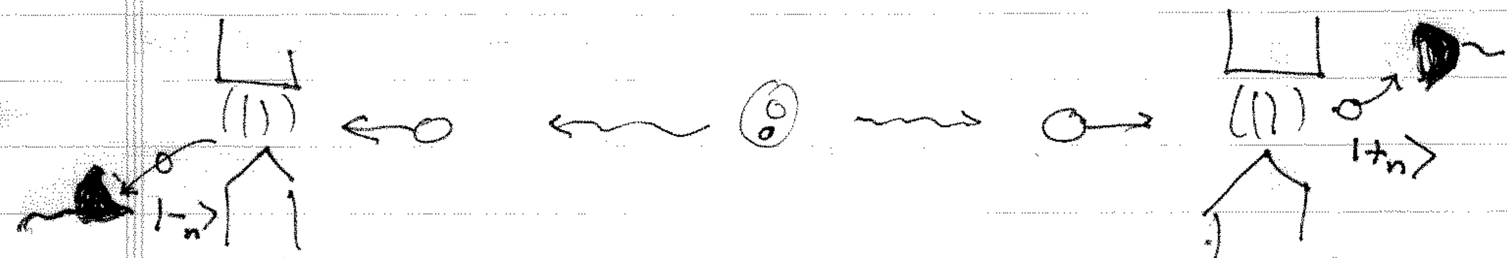
Which can be seen by a rotation operator to $|\Psi\rangle$, or by direct substitution

$$|+_n\rangle = \cos\frac{\theta}{2}|+_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-_z\rangle$$
$$|-_n\rangle = \sin\frac{\theta}{2}|+_z\rangle - e^{i\phi}\cos\frac{\theta}{2}|-_z\rangle$$

where θ, ϕ are the polar angles defining $\vec{e}_n$.

The up-shot of this is that the measurements of spin directions for the two particles are anti-correlated no matter what direction $\vec{e}_n$ we choose to measure along.

We imagine then a situation in which two particles are bound in a spin singlet and then decay into two daughters. One spin goes to Alice the other to Bob. They can measure the direction of the spin using a Stern-Gerlach device, with the gradient field along any axis. If they choose the same axis the spin directions are anticorrelated



50% probability of the above

The EPR argument goes through as before. If Alice orients her S-G apparatus along $\vec{e}_z$ she can predict the value of $\hat{\sigma}_z$ for Bob, without disturbance by a signal traveling at the speed of light. If Alice measure along $\vec{e}_x$, she predicts $\hat{\sigma}_x$ for Bob. EPR would say both had to have been objectly real. But $\hat{\sigma}_x$ and $\hat{\sigma}_z$ don't commute $\Rightarrow$ Q.M. is incomplete.

Bell's Inequalities (Physics 1 195 (1964))

John Bell returned to reëxamine the EPR paradox and the question of hidden variables. He knew that there was at least one hidden variable theory the reproduced Q.M. due to David Bohm

Aside: "Bohmian" Q.M. (Phys. Rev. 85 166 (1952))

Recall that the Schrödinger equation

$$\frac{\hbar}{-i}\frac{\partial \psi(\vec{x},t)}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi$$

can be writen in term of polar form $\psi = \sqrt{\rho}\, e^{iS/\hbar}$

where $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

$$\frac{\partial S}{\partial t} + \frac{|\vec{\nabla} S|^2}{2m} + V(\vec{x}) - \frac{\hbar^2}{2m\sqrt{\rho}}\nabla^2(\sqrt{\rho}) = 0$$

with $\vec{J} = \rho \frac{\vec{\nabla} S}{m}$

The second equation can be interpreted as a classical Hamilton-Jacobi equation, but with a new potent

$$U(\vec{x}) = V(\vec{x}) + V_Q(\vec{x})$$

$$V_Q(\vec{x}) = -\frac{\hbar^2}{2m\sqrt{\rho}}\nabla^2(\sqrt{\rho})$$ = "Quantum Potential"

Particles then follow deterministic trajectories governed by a Newton's law. Note the $V_Q(\vec{x})$ depends on the wave function.

This is a "hidden variable" that effects the trajectory, but which we do not have access to.

Now for a two-body case

$$U(\vec{x}_1, \vec{x}_2) = V(\vec{x}_1, \vec{x}_2) + V_Q(\vec{x}_1, \vec{x}_2)$$

$$\text{where } V_Q(\vec{x}_1, \vec{x}_2) = -\frac{\hbar^2}{2m\sqrt{\rho(\vec{x}_1,\vec{x}_2)}}\left[\nabla_1^2 + \nabla_2^2\right]\sqrt{\rho(\vec{x}_1,\vec{x}_2)}$$

$$\rho(\vec{x}_1, \vec{x}_2) = |\Psi(\vec{x}_1, \vec{x}_2)|^2$$

In general, the quantum potential will be <u>nonlocal</u> since the wave function is. Thus, the trajectory of particle 2 will effect particle 1 even if they are not within each other's light cones.

This is of course, nonrelativistic Q.M., so perhaps we should not expect it to ~~be~~ adhere to the law of relativity. No relativistic generalization of the Bohm theory has yet been produced.

Back to Bell:

Bell knew that the Bohm theory was a hidden variable model of Q.M., but is was manefestly nonlocal. He thus asked whether this was a requirement of any hidden variable model. This would be in contradiction to EPR who argued, from locality and counterfactual logic, that a local hidden variable model (LHVM) could reproduce the results of quantum measurement and furthermore the Q.M. was "incomplete".

Consider then the Bohm version of EPR. Alice and Bohm make measurements of $\hat{\sigma}_n$ which has eigenvalues ± 1. A ~~local~~ LHVM assigns an objective value $+1$ or -1 to this observale, which depends on the axis $\vec{e}_n$ and the hidden variables, we denote collectively as λ. thus:

Alice finds $\quad A(\vec{e}_a, \lambda)$

Bob finds $\quad B(\vec{e}_b, \lambda)$

Note the value A does not depend on $\vec{e}_b$

B does not depend on $\vec{e}_a$

This is the statement of locality.

Bell then considered average values which arise from our uncertain knowledge of the hidden variables λ. We assign a probability distribution $P(\lambda)$. Thus, for example

$$C(a,b) \equiv \langle \sigma_a \, \sigma_b \rangle_{LHVM} = \int d\lambda \, P(\lambda) \, A(\vec{e}_a, \lambda) \, B(\vec{e}_b, \lambda)$$

The LHVM is supposed to reproduce the Q.M. expectation value

$$C(a,b) \equiv \langle \hat{\sigma}_a \hat{\sigma}_b \rangle_{QM} = \langle \Psi | \hat{\sigma}_a^{(A)} \otimes \hat{\sigma}_b^{(B)} | \Psi \rangle$$

For the case of the singlet, you can show

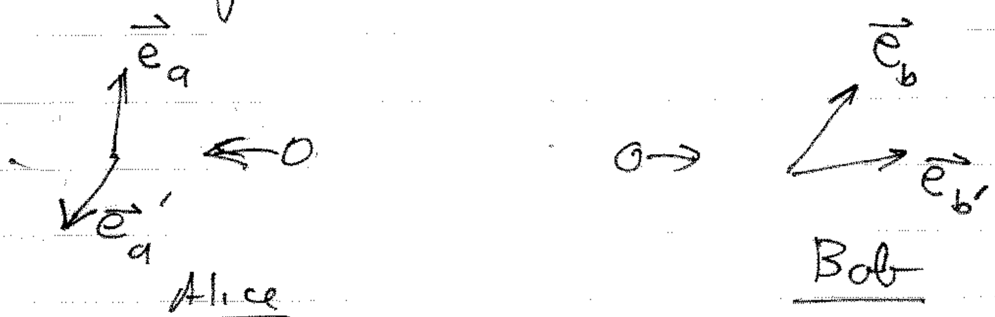
$$\langle \hat{\sigma}_a \hat{\sigma}_b \rangle_{QM} = - \vec{e}_a \cdot \vec{e}_b$$

Bell showed that the correlation functions arising from a LHVM are constrained by inequalities. We we see that the QM correlation ~~is~~ violates this inequality and the Q.M. cannot be described by a local hidden variable model.

<u>CHSH:</u> Phys. Rev Lett <u>23</u> 880 (1969).

The form of the Bell inequality we consider here was due to Clauser - Holt - Shimony - Horne ~~(?)~~ (CHSH). It is a generalization of the one originally derived by Bell, but more amenable to experiments.

Consider a scenario in which Alice measures her spin along <u>either</u> $\vec{e}_a$ or $\vec{e}_{a'}$, Bob along $\vec{e}_b$ or $\vec{e}_{b'}$



Define the observable

$$\hat{S} = \hat{\sigma}_a \hat{\sigma}_{b'} - \hat{\sigma}_a \hat{\sigma}_b + \hat{\sigma}_{a'} \hat{\sigma}_{b'} + \hat{\sigma}_{a'} \hat{\sigma}_b$$
$$= \hat{\sigma}_a (\hat{\sigma}_{b'} - \hat{\sigma}_b) + \hat{\sigma}_{a'} (\hat{\sigma}_{b'} + \hat{\sigma}_b)$$
$$\Rightarrow \langle \hat{S} \rangle = C(a,b') - C(a,b) + C(a',b') + C(a',b)$$

Consider the values of S is a LHVM

$$S_\lambda = A(\vec{e}_a,\lambda)\left[B(\vec{e}_{b'},\lambda) - B(\vec{e}_b,\lambda)\right]$$
$$+ A(\vec{e}_{a'},\lambda)\left[B(\vec{e}_{b'},\lambda) + B(\vec{e}_b,\lambda)\right]$$

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All measurements are "objectively real" and ~~takes~~ takes on values $+1$ or -1 with some probability determined by λ. We see that if $B(\vec{e}_{b'}, \lambda)$ is ~~the~~ same as $B(\vec{e}_b, \lambda)$ the first term vanishes and

$S_\lambda = 2A(\vec{e}_{a'}, \lambda) B(\vec{e}_b, \lambda)$. If the B values are opposite $S_\lambda = -2 A(\vec{e}_a, \lambda) B(\vec{e}_b, \lambda)$.

Thus we see that in a LHVM $S_\lambda = +2$ or -2 in any run of the experiment.

$$\therefore \langle S \rangle_{LHVM} = \int d\lambda \, P(\lambda) \, S_\lambda \quad \text{is bounded}$$

$$\boxed{-2 \leq \langle S \rangle_{LHVM} \leq 2}$$

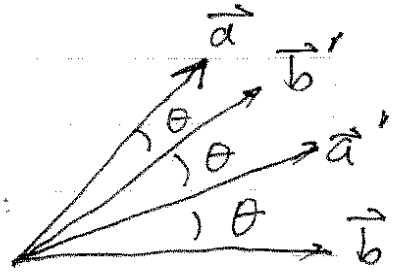
This is the CHSH Bell inequality

What does Q.M. give?

$$\langle \hat{S} \rangle_{QM} = -(\vec{a} \cdot \vec{b}' - \vec{a} \cdot \vec{b} + \vec{a}' \cdot \vec{b}' + \vec{a}' \cdot \vec{b})$$

Does this satisfy Bell's inequality?

Consider a set of possible measurement directions:

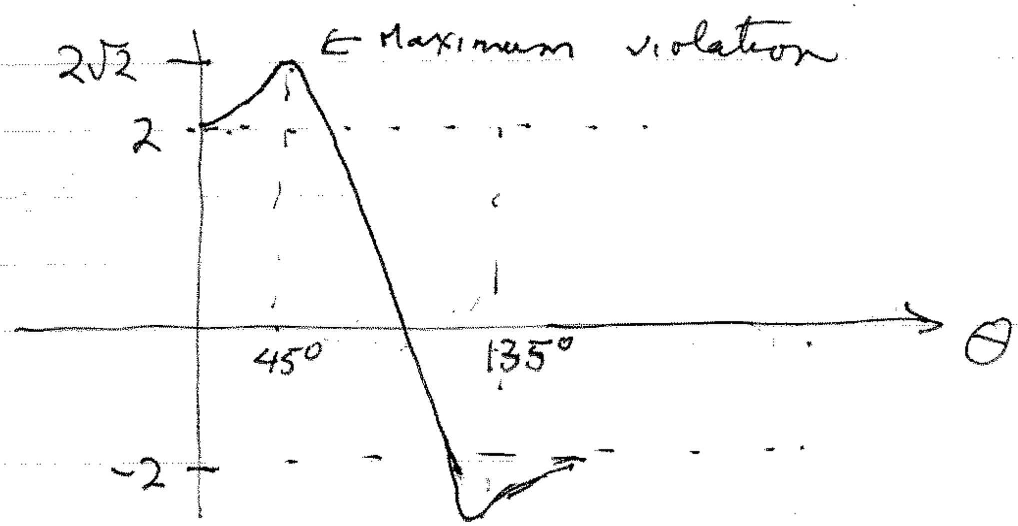


$$\Rightarrow \langle \hat{S} \rangle_{Q.M.} = 3\cos\theta - \cos 3\theta$$

For small θ $\quad \langle \hat{S} \rangle_{QM} \approx 3\left(1 - \frac{\theta^2}{2}\right) - \left(1 - \frac{9}{2}\theta^2\right)$

$$= 2 + 3\theta^2$$

$\Rightarrow$ $\langle \hat{S} \rangle_{Q.M.} > 2 \Rightarrow$ Quantum Mechanics Violates Bell inequality



Quantum entanglement <u>cannot</u> be captured by a L.H.V.M.

The upshot of this is that, contrary to EPR's assumption, a L.H.V.M. cannot reproduce all of quantum theory. If we want to devise a hidden variable model, it must be non local, as in the Bohmian theory. Alternatively, we must content ourselves with the fact that the microscopic properties are <u>not</u> <u>objectly</u> <u>real</u>. We find a random result correlated with the measurement apparatus as Bohr described. Only the <u>whole system</u>, micro plus macro apparatus is physically objective.

<u>Experiments</u>:

There have been numerous experiments that have shown violations of a Bell inequality in agreement with Q.M. Most have been done with photons, measuring the polarization (a two state system which maps onto spin 1/2). Notable examples are:

- Kwiat et al ~~PR~~ PRL <u>60</u> 773 (1992)
- Freedman + Clauser: PRL <u>28</u> 938 (1972) (The first try)
- Aspect et al: PRL <u>47</u> 460 (1981) (The first observed violation of Bell)