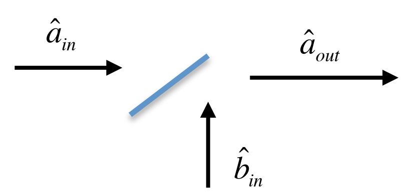
Physics 581, Quantum Optics II
Problem Set #2
Due: Tuesday September 27, 2016 @ 5PM

Problem 1: Shot-noise and vacuum fluctuations (25 Points)

Squeezed states are not robust because of photon absorption (loss). There are many ways to understand this. Squeezing is associated with photon pair-correlations. Loss will randomly remove a photon, and not both in the correlated pair. The remaining photon will then just add shot noise. Another way to understand this is from the perspective of the continuous variables.

Classically, linear loss results in attenuation of the field amplitude $E \to e^{-\kappa} E = \eta E$, where η is the loss coefficient. We can model this attenuation by a partial transmitting beam splitter, with transmission amplitude η. Quantumly, we cannot make the transformation, $\hat{a} \to \eta \hat{a}$, because the commutation relations are not preserved. Stated in another way, *we cannot attenuate the vacuum fluctuations*.

(a) Consider a linear transformation between two modes at a beam splitter with transmission amplitude η

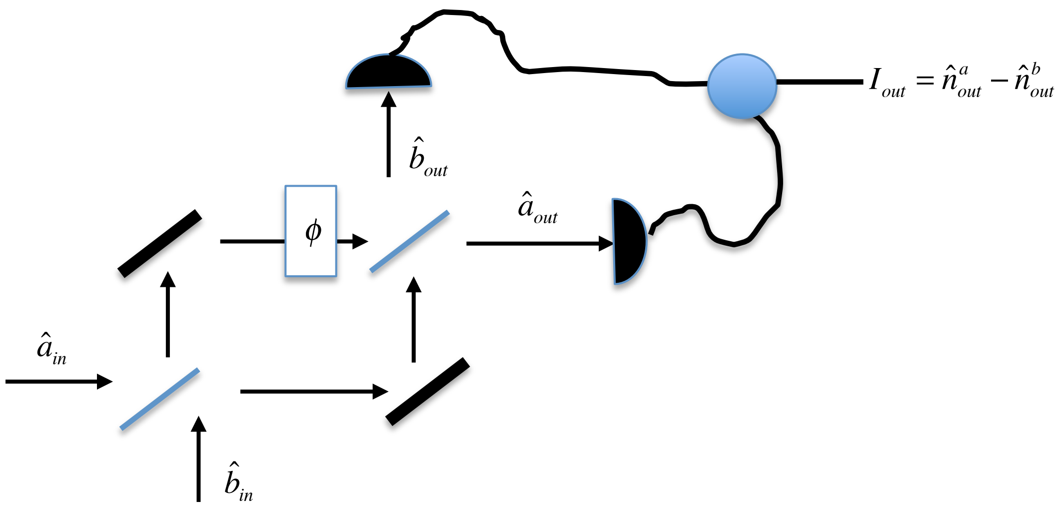


Show that if the vacuum enters in mode-b, the output quadrature fluctuations in mode-a are

$$\Delta X^2_{\Theta,out} = |\eta|^2 \Delta X^2_{\Theta,in} + \left(1 - |\eta|^2\right)\frac{1}{2}.$$

Interpret this result and show how squeezing is degraded by photon loss.

(b) Consider now a Mach-Zender interferometer



Show that output photocurrent operator corresponding to the difference of the photocounts at the two output ports is

$$\hat{I}_{out} = \cos\phi\left(\hat{n}^a_{in} - \hat{n}^b_{in}\right) - \sin\phi\left(\hat{a}^\dagger_{in}\hat{b}_{in} + \hat{b}^\dagger_{in}\hat{a}_{in}\right).$$

(c) If the input signal enters through port-*a* and vacuum in port-*b*, then the show the mean output signal and fluctuations are

$$\left\langle \hat{I}_{out} \right\rangle = \cos\phi\left\langle \hat{n}^a_{in} \right\rangle, \ \left\langle \Delta\hat{I}^2_{out} \right\rangle = \cos^2\phi\left\langle (\Delta\hat{n}^a_{in})^2 \right\rangle + \sin^2\phi\left\langle \hat{n}^a_{in} \right\rangle$$

(d) The mean output signal is what we know as the interference "fringes." The fluctuations determine the noise. The term $\cos\phi\left\langle \Delta\hat{n}^{a\,2}_{in} \right\rangle$ in $\left\langle \Delta\hat{I}^2_{out} \right\rangle$ is the contribution of the noise from the input signal. Show that the term $\sin\phi\left\langle \hat{n}^a_{in} \right\rangle$ arises solely do to the vacuum fluctuations entering from the unused port-*b*.

(e) In an application such as LIGO, which measures a tiny effect such as a gravity wave, we typically operate near a node of the fringe so that we are measuring an effect away from zero. Thus we set $\phi = \pi/2 + \delta\phi$, where $\delta\phi << 1$ depends on the strength of the gravity wave. In that case, show that the signal-to-noise ratio (SNR) is

$$SNR = \frac{\left\langle \hat{I}_{out} \right\rangle}{\sqrt{\left\langle \Delta\hat{I}^2_{out} \right\rangle}} \approx \delta\phi\sqrt{\left\langle \hat{n}^a_{in} \right\rangle}.$$

This is known as the "standard quantum limit" and is limited solely by the vacuum noise vacuum noise entering port-b.

(f) The seminal work on C. M. Caves, Phys. Rev. D **23**, 1693 (1981), showed that we could improve the *SNR* by injecting *squeezed vacuum* into the unused port. Show that that in this case, the *SNR* is

$$SNR = \frac{\langle \hat{I}_{out} \rangle}{\sqrt{\langle \Delta \hat{I}^2_{out} \rangle}} \approx e^r \delta\phi \sqrt{\langle \hat{n}^a_{in} \rangle},$$

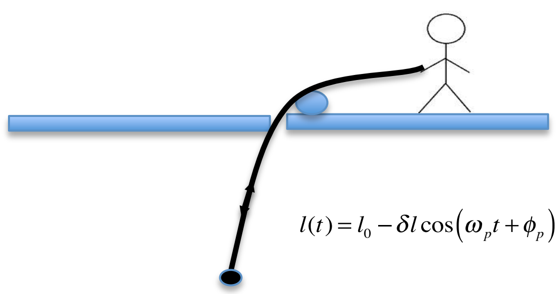
where r is the squeezing parameter. This is currently be implemented in the latest generation of LIGO.

http://www.nature.com/nphoton/journal/v7/n8/full/nphoton.2013.177.html

Problem 2: Parametric resonance (15 points)

Optical parametric amplification that we studied in the context of three-wave mixing in nonlinear optics is a general phenomenon in nonlinear dynamics know as *parametric resonance*. We study the basic problem here.

Consider an oscillator with a *time-dependent frequency* $\ddot{x} + \omega^2(t)x = 0$. For example, consider a pendulum whose length is periodically modulated



When $\delta l << l_0$ the pendulum oscillations satisfy the Mathieu equation

$$\ddot{x} + \left(\omega_0^2 + \varepsilon \cos(\omega_p t - \phi_p)\right)x = 0, \text{ where } \varepsilon = \frac{\delta l}{l_0}\omega_0^2 \text{ and } \omega_0 = \sqrt{g / l_0}$$

There is no general analytic solution to this problem. We can, however, solve this approximately. Our goal is to show that there a nonlinear resonance, at which point we exponential pump energy into the system.

(a) Write the general solution as $x = \mathrm{Re}\left(\alpha(t) e^{-i\omega_0 t}\right)$. For weak driving take $|\dot{\alpha}| << \omega_0 |\alpha|$.

Show that

$$\dot{\alpha}-\dot{\alpha}^{*}e^{2i\omega_0 t}\approx -i\frac{\varepsilon}{4\omega_0}\left(\alpha e^{i\phi_P}e^{-i\omega_p t}+\alpha e^{-i\phi_P}e^{+i\omega_p t}+\alpha^{*}e^{-i\phi_P}e^{+i(\omega_P+2\omega_0)t}+\alpha^{*}e^{i\phi_P}e^{-i(\omega_P-2\omega_0)t}\right).$$

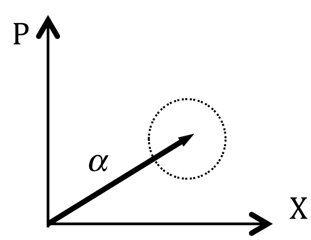
(b) This equation shows "parametric resonance" when $\omega_P = 2\omega_0$ (the term "parametric" comes from the idea that we were modulating a parameter in the original oscillator). Ignoring the rapidly oscillating term, show that at parametric resonance,

$$\dot{\alpha}\approx\zeta\alpha^{*},\ \ \zeta = re^{i2\theta}\ \text{ with solution } X_\theta(t)=e^{-r}X_\theta(0),\ \ P_\theta(t)=e^{+r}P_\theta(0),\ \ \alpha e^{-i\theta}\equiv X_\theta+iP_\theta$$

(find ζ, r, θ)

We are all familiar with this phenomenon. As a child on a swing, we pump our legs back and forth, effectively increasing or decreasing the length of the pendulum. If we pump at twice the natural frequency we amplify our motion, but only if we pump at the right phase! In nonlinear optics, the pump laser effectively changes the optical path length of the signal and can thus parametrically amplify the signal.

(c) Parametric resonance leads to *phase-sensitive amplification*. Consider a classical statistical distribution of initial complex amplitudes.



For the conditions such that $\theta = 0$, sketch the resulting output distribution. Comment.

Problem 3: Squeezing in the Heisenberg Picture (20 Points)

(a) Consider a "pure squeezing" Hamiltonian, $\hat{H} = \hbar\kappa^* \hat{a}^2 + \hbar\kappa \hat{a}^{\dagger 2}$. Show that the Heisenberg equations of motion and solutions are

$$\frac{d\hat{a}}{dt} = -i2\kappa \hat{a}^\dagger,\ \frac{d\hat{a}^\dagger}{dt} = +i2\kappa^* \hat{a} \Rightarrow \hat{a}(t) = \cosh(r)\hat{a}(0) - e^{i2\theta}\sinh(r)\hat{a}^\dagger(0).\ \text{Find } r, \theta.$$

This is the Bogoliubov transformation, corresponding to parametric amplification.
Note: $\hat{a}(t) = \mu\hat{a}(0) - \nu\hat{a}^\dagger(0)$, with $|\mu|^2 - |\nu|^2 = 1$, and thus $r = \sinh^{-1}(|\nu|)$.

(b) Now consider a squeezing interaction in the present of a rotation (caused by a phase mismatch), $\hat{H} = \hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\kappa^*\hat{a}^2 + \hbar\kappa\hat{a}^{\dagger 2}$. Find the Heisenberg equations of motion and show that the solution is

$$\hat{a}(t) = \left[\cosh\Omega - i\frac{\Delta t}{\Omega}\sinh\Omega\right]\hat{a}(0) - \left[e^{i2\theta}\frac{r}{\Omega}\sinh\Omega\right]\hat{a}^\dagger(0)$$

where r, θ are as in part (a) and $\Omega = \sqrt{r^2 - (\Delta t)^2}$.

(c) Show that, as in part (a), in part (b), $\hat{a}(t) = \mu\hat{a}(0) - \nu\hat{a}^\dagger(0)$, with $|\mu|^2 - |\nu|^2 = 1$. This is generic result for any Hamiltonian that is quadratic in $\hat{a}, \hat{a}^\dagger$ (more on this fact to come). Using this, show that this interaction leads to squeezing with squeezing parameter

$$\tilde{r} = \sinh^{-1}\left(\frac{r}{\Omega}\sinh\Omega\right) = \log\left(\frac{r}{\Omega}\sinh\Omega + \sqrt{1 + \left(\frac{r}{\Omega}\sinh\Omega\right)^2}\right)$$

(d) Show that only when we have perfect phase matching ($\Delta = 0$) do we achieve exponential growth with time (amplification) of one quadrature and deamplifaction (squeezing) of the other.

Problem 4: Some more boson Algebra (20 Points – extra credit)

(a) Show that the displacement operators are orthogonal according to the Hilbert-Schmidt inner product, $Tr\left(\hat{D}^{\dagger}(\alpha)\hat{D}(\beta)\right)=\pi\delta^{(2)}(\alpha-\beta)$.

Hint: Recall $Tr\left(\hat{A}\right)=Tr\left(\int\frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|\hat{A}\right)=\int\frac{d^2\alpha}{\pi}\langle\alpha|\hat{A}|\alpha\rangle$

(b) We have shown that the Fourier transform of the displacement operators are

$$\hat{T}_{\sigma}(\alpha)\equiv\int\frac{d^2\beta}{\pi}\hat{D}_{\sigma}(\beta)e^{\alpha\beta^*-\alpha^*\beta}=\pi\left\{\delta^{(2)}(\alpha-\hat{a})\delta^{(2)}(\alpha^*-\hat{a}^{\dagger})\right\}_{\sigma}.$$

Show that $\hat{T}_{-1}(\alpha)=|\alpha\rangle\langle\alpha|$ (Hint: Insert $\int\frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|$ appropriately)

(c) Show that for a pure state $\hat{\rho}=|\psi\rangle\langle\psi|$, the Wigner function is

$$W(X,P)=\int_{-\infty}^{\infty}\frac{dY}{2\pi}\psi^*\left(X+\frac{Y}{2}\right)\psi^*\left(X-\frac{Y}{2}\right)e^{-iPY} \text{ , where } W(X,P)=\frac{1}{2}W(\alpha) \ .$$

(d) Show that the Wigner function yields the correct marginals in X and P,

$$\int_{-\infty}^{\infty}dP\,W(X,P)=|\psi(X)|^2,\quad \int_{-\infty}^{\infty}dX\,W(X,P)=|\tilde{\psi}(P)|^2,$$

and for an arbitrary quadrature

$$\int_{-\infty}^{\infty}dP_{\theta}\,W(X,P)=|\tilde{\psi}(X_{\theta})|^2$$