**Physics 581, Quantum Optics II**

**Problem Set #2**

**Due: Tuesday October 11, 2016 @ 5PM**

**Problem 1: Nonclassical light generation via the Kerr effect. (15 points)**

In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\chi^{(3)}}{2} :\hat{I}^2: = \frac{\hbar\chi^{(3)}}{2}\hat{a}^{\dagger 2}\hat{a}^2 .$$

(a) Suppose we inject a strong coherent state into a nonlinear fiber with Kerr response. *Linearize* this Hamiltonian about the mean field via the substitution $\hat{a} = \alpha + \hat{b}$, and keep terms only up to quadratic order in $\hat{b}$ and $\hat{b}^\dagger$. Show that the resulting leads to squeezing.

(b) Now let's go beyond the linear approximation. Show that for a long time such that $\chi^{(3)}t = \pi$, the state becomes a Schrödinger cat, $\left(e^{i\pi/4}|-i\alpha\rangle + e^{-i\pi/4}|i\alpha\rangle\right)/\sqrt{2}$.

**Problem 2: Calculation of some quasiprobability functions (25 points)**

(a) Find the *P. Q,* and *W* distributions for a thermal state

$$\hat{\rho} = \frac{e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_B T}}{Z}, Z = Tr(e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_B T}) = \text{partition function}$$

and show they are *Gaussian* functions. For example, you should find $P(\alpha) = \frac{1}{\pi\langle n\rangle}\exp\left(-\frac{|\alpha|^2}{\langle n\rangle}\right)$. Show that these three distributions give the proper functions in the limit, $\langle n\rangle \to 0$, i.e. the vacuum.

(b) Find the *P. Q,* and *W* distributions squeezed state $|\psi\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$ . In what sense is this state nonclassical?

(c) Find the Glauber-Sudharshan P-representation for a Fock state $|\psi\rangle = |n\rangle$. Comment.

(d) Consider a superposition state of two "macroscopically" distinguishable coherent states,

$|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, $|\alpha_1 - \alpha_2| >> 1$, where $N = \left[2\left(1 + \exp\{-|\alpha_1 - \alpha_2|^2\}\right)\right]^{-1/2}$ is normalization.

This state is often referred to as a "Schrodinger cat", and is very nonclassical. Calculate the Wigner function, for the case $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$, with α real, and plot it for different values of $|\alpha_1 - \alpha_2| = 2\alpha$. Comment please.

(e) Calculate the marginals of the Schrödinger-cat Wigner function in *X* and *P* and show they are what you expect.

**Problem 3: An Alternative Representation of the Wigner Function. (20 points)**

We have shown that Wigner function could be expressed as

$$W(\alpha) = \frac{1}{\pi} Tr\left(\hat{\rho}\hat{T}(\alpha)\right) = \frac{1}{\pi}\left\langle \hat{T}(\alpha)\right\rangle \text{, where } \hat{T}(\alpha) = \int \frac{d^2\beta}{\pi} \hat{D}(\beta) e^{\alpha\beta^* - \beta^*\alpha}$$

(a) Show that $\hat{T}(\alpha) = \hat{D}(\alpha)\hat{T}(0)\hat{D}^\dagger(\alpha)$.

(b) Show that $\hat{T}(0) = 2(-1)^{\hat{a}^\dagger\hat{a}}$. (This is a tough problem. You may assume the answer and work backwards or try to find a direct proof).

Note: the operator $(-1)^{\hat{a}^\dagger\hat{a}} = \sum_n (-1)^n |n\rangle\langle n| = \int dX |-X\rangle\langle X|$ is the "parity operator" (+1 for even parity, -1 for odd parity). Thus we see that the Wigner function at the origin is given by the expected value of the parity.

$$W(0) = \frac{2}{\pi} Tr\left[\hat{\rho}(-1)^{\hat{a}^\dagger\hat{a}}\right] = \frac{2}{\pi}\sum_n (-1)^n \langle n|\hat{\rho}|n\rangle .$$

(c) Show that general expression

$$\hat{T}(\alpha) = 2\hat{D}(\alpha)(-1)^{\hat{a}^\dagger\hat{a}}\hat{D}^\dagger(\alpha) = 2\sum_n (-1)^n \hat{D}(\alpha)|n\rangle\langle n|\hat{D}^\dagger(\alpha) ,$$

$$\text{and thus } W(\alpha) = \frac{2}{\pi}\sum_n (-1)^n \langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle .$$

This expression provides a way to "measure" the Wigner function. One displaces the state to the point of interest, $\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)$, one then measures the photon statistics $p_{n\alpha} = \langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle$. Putting this in the parity sum gives $W(\alpha)$ at that point!

This is a form a quantum-state reconstruction, also know as "quantum tomography," which we will study in a future problem.

**Problem 4: Toward an optical "Schrödinger cat state." (20 points)**

Creating a "Schrödinger cat state," e.g. $\left|cat_\phi(\alpha_0)\right\rangle = N\left(\left|\alpha_0\right\rangle + e^{i\phi}\left|-\alpha_0\right\rangle\right)$, where the normalization $N = 1/\sqrt{2\left(1+\cos\phi\, e^{-2|\alpha|^2}\right)}$, is a challenging task in the optical regime because we do not have sufficient nonlinearity with low loss (in the microwave regime, cavity and circuit QED has achieved this – more on that in another problem). Producing something close to such a state for applications in Quantum Information Processing has been an important goal.

Consider a squeezed single photon Fock state, $\left|r,1\right\rangle \equiv \hat{S}(r)\left|1\right\rangle$

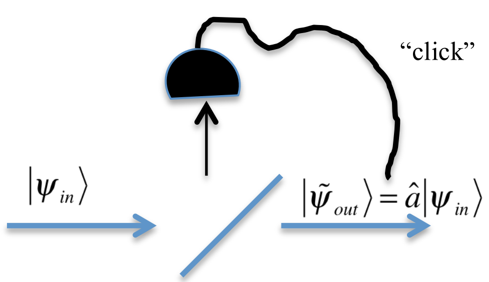
(a) Show that the Wigner function of this state is $W(\alpha) = -\frac{1}{\pi^2} e^{-2|b|^2} L_1(4|b|^2)$ where $b = \alpha^*\cosh r + \alpha\sinh r$ and $L_1$ is the first-order Laguerre polynomial. Plot $W$.

(b) Show that the fidelity between the "odd" cat-state and the squeezed Fock state is

$$F(r,\alpha_0,\pi) \equiv \left|\left\langle cat_\pi(\alpha_0)\middle|r,1\right\rangle\right|^2 = \frac{2\alpha_0^2\exp\left[\alpha_0^2\left(\tanh r - 1\right)\right]}{(\cosh r)^3\left(1-\exp[-2\alpha_0^2]\right)} \text{ (where } \alpha_0 \text{ is real).}$$

(c) Make a surface plot of $F$ as a function of $r$ and $\alpha_0$. Under what parameters can the squeezed Fock state well approximate the cat state?

While the squeezed Fock state can approach cat state, squeezing a single photon state is not easy to achieve either. The output of a nonlinear optical processes is typically a squeezed vacuum. This is a Gaussian state, which is classical if we perform only homodyne measurements. However, if we have access to other resources, such a photon counting, we can transform this into a non-Gaussian, fully quantum resource. Consider the following experiment:




The light is incident on a highly transmitting beam splitter. Rarely one photon is reflected and detected. Conditioned on that "click," the output state has one of the photons annihilated. This state is "post selected" and the probability of producing it is rare. Nonetheless, this is a highly non-Gaussian operation.

The state produced is a "photon subtracted squeezed state." This operation is non-Unitary, so the post-measurement state is $\left|\psi_{out}\right\rangle = \hat{a}\hat{S}(r)\left|0\right\rangle / \left\|\hat{a}\hat{S}(r)\left|0\right\rangle\right\|$.

(d) Show that $\left|\psi_{out}\right\rangle = \hat{S}(r)\left|1\right\rangle$, the squeezed Fock state.