Physics 581, Quantum Optics II
Problem Set #4
Due: Tuesday November 1, 2016

Problem 3: The EPR state (30 points)
The Einstein-Podolsky-Rosen (EPR) paradox is based around a thought experiment of measurements on an entangled state of the motion of two particles. The EPR state is a simultaneous eigenstate of relative position and the center-of-mass momentum

$$\left(\hat{X}_A - \hat{X}_B\right)\left|EPR\right\rangle = X_{rel}\left|EPR\right\rangle \ , \ \left(\hat{P}_A + \hat{P}_B\right)\left|EPR\right\rangle = P_{com}\left|EPR\right\rangle$$

The purpose of this problem is to show how one can create an approximation to this state in quantum optics, and to study their entanglement properties.

(a) We showed that the photon pair produced in spontaneous parametric down conversion was entangled in frequency and time of emission. By selecting a narrow pinhole of phase-matched signal (*s*) and idler (*i*) directions, the state can be written

$$\left|\Psi\right\rangle = \int d\omega_s\, \tilde{A}(\omega_s)\left|\omega_s\right\rangle_s \otimes \left|\omega_p - \omega_s\right\rangle_i = \int dt'\, A(t-t')\left|t\right\rangle_s \otimes \left|t'\right\rangle_i .$$

Here, $\tilde{A}(\omega_s)$ is the spectrum of signal frequencies allowed through the pinhole, $\left|\omega\right\rangle_{s(i)}$ is a mode with frequency ω travelling in the $s(i)$ direction, and $\left|t\right\rangle_{s(i)}$ is a "temporal mode" representing a photon localized near position ct along the beam.

Argue that in the limit, $\tilde{A}(\omega_s) \to 1/2\pi$ *,* $A(t-t') \to \delta(t-t')$*, this is the EPR state. What plays the role of position and momentum?*

Consider now a parametric oscillator beyond the perturbative limit, where two modes (*A* and *B*) are phase matched with the pump. The resulting output state is a two-mode squeezed vacuum state

$$\left|0,0\right\rangle_r = \hat{S}_{AB}(r)\left|0\right\rangle_A\left|0\right\rangle_B = e^{r\left(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b}\right)}\left|0\right\rangle_A\left|0\right\rangle_B$$

Our goal is to show that in the limit of infinite squeezing, the is the EPR state.

(b) Show that, $\hat{S}^\dagger_{AB}\left(\hat{P}_A \pm \hat{P}_B\right)\hat{S}_{AB} = \left(\hat{P}_A \pm \hat{P}_B\right)e^{\mp r}$, and thus this operation squeezes the "relative position" and "center-of-mass momentum" quadratures.

(c) Show that $\hat{S}^\dagger_{AB}\hat{X}_A\hat{S}_{AB} = \cosh r\hat{X}_A + \sinh r\hat{X}_B$, $\hat{S}^\dagger_{AB}\hat{X}_B\hat{S}_{AB} = \cosh r\hat{X}_B + \sinh r\hat{X}_A$. This is a Heisenberg statement.

(d) From this argue that, up to normalization (which is tricky position)

$$\hat{S}_{AB}(r)\left|X_A\right\rangle_A\left|X_B\right\rangle_B = \left|\cosh rX_A + \sinh rX_B\right\rangle_A\left|\cosh rX_B + \sinh rX_A\right\rangle_B$$

(e) Show that the (normalized) position space wave function for the two modes is

$$\Psi_r(X_A,X_B) = \langle X_A|\langle X_B||0,0\rangle_r = \frac{1}{\sqrt{\pi}}e^{-\frac{(X_A-X_B)^2}{4e^{-2r}}}e^{-\frac{(X_A+X_B)^2}{4e^{+2r}}} \quad \text{(plot for } r\text{=2)}$$

and in the limit of infinite squeezing $\lim_{r\to\infty}\Psi_r(X_A,X_B) \Rightarrow \delta(X_A - X_B)$

(f) By similar arguments, show that the (normalized) momentum space wave function is

$$\tilde{\Psi}_r(P_A,P_B) = \langle P_A|\langle P_B||0,0\rangle_r = \frac{1}{\sqrt{\pi}}e^{-\frac{(P_A+P_B)^2}{4e^{-2r}}}e^{-\frac{(P_A-P_B)^2}{4e^{+2r}}} \quad \text{(plot for } r\text{=2)}$$

and in the limit of infinite squeezing $\lim_{r\to\infty}\tilde{\Psi}_r(P_A,P_B) \Rightarrow \delta(P_A + P_B)$

Thus argue that in the limit of infinite squeezing, the two-mode squeezed vacuum is the EPR state.

(g) Show that in the limit of infinite squeezing, the two-mode squeezed state can be expressed as

$$\lim_{r\to\infty}|0,0\rangle_r \Rightarrow |EPR\rangle = \int dX\,|X\rangle_A \otimes |X\rangle_B = \int dP\,|P\rangle_A \otimes |-P\rangle_B = \sum_n |n\rangle_A \otimes |n\rangle_B$$

Note: This is maximally entangled state in infinite dimensions. It is not a physical state, however, as it requires infinite energy. Nonetheless, we approximate is with large, but finite squeezing.

(h) Show that the Wigner function for the two-mode state is

$$W(X_A,P_A,X_B,P_B) = \left|\Psi_r(X_A,X_B)\right|^2\left|\tilde{\Psi}_r(P_A,P_B)\right|^2 = \frac{1}{\pi^2}e^{-\frac{(X_A-X_B)^2+(P_A+P_B)^2}{2e^{-2r}}}e^{-\frac{(X_A+X_B)^2+(P_A-P_B)^2}{2e^{2r}}}$$

(i) Extra credit: The Wigner function is positive, meaning there is a classical local probabilistic description of joint measurements of X_A, X_B, P_A, P_B. What are the implications for the EPR paradox and Bell's inequalities?

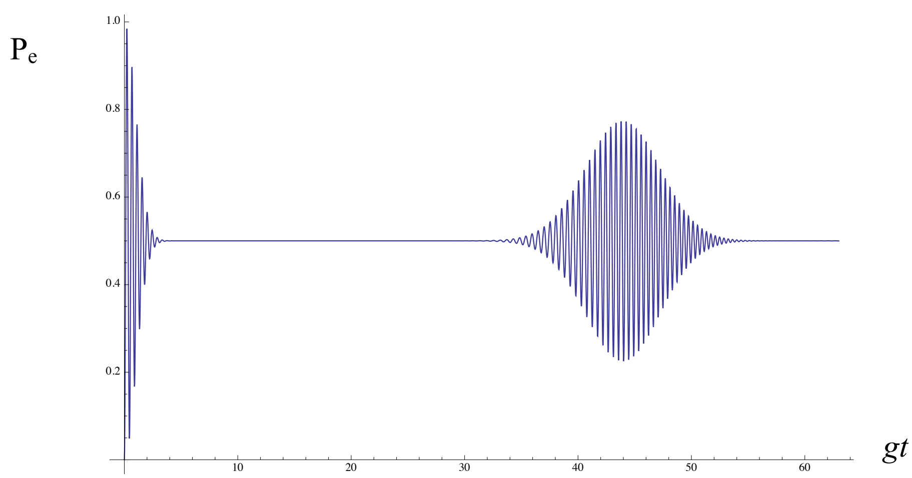
Problem 2: Entanglement and the Jaynes-Cummings Model (30 points)

One the most fundamental paradigms in quantum optics is the coupling of a two-level atom to a single mode of the quantized electromagnetic field. In the rotating wave approximation, this is governed by the Jaynes-Cummings model (JCM),

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_0 \frac{\hat{\sigma}_z}{2} + \hbar g\left(\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-\right).$$

This is a bipartite system with tensor product Hilbert space for the atom and field, $\mathcal{H}_{AF} = \mathscr{h}_A \otimes \mathscr{h}_F$, where $\mathscr{h}_A$ is the two-dimensional Hilbert space of the two-level atom, and $\mathscr{h}_F$ is the infinite dimensional Hilbert space of the harmonic oscillator that describes the mode. The goal of this problem is to understand the entanglement between the atom and mode, generated by the JCM.

Last semester, we studied how this leads to collapse and revival of Rabi oscillation that follows from an initial product state with the field in a coherent state and the atom in, e.g., the ground state $|\Psi(0)\rangle_{AF} = |g\rangle_A \otimes |\alpha\rangle_F$. The probability to find the atom in the excited state oscillates as shown (here for $\langle n \rangle = |\alpha|^2 = 49$)



The collapse is due to the variation of the quantum Rabi oscillations with different number; the revival is uniquely a quantum effect arising from the discreteness of the quantized field, occurring at a time $gt_r \approx 2\pi\sqrt{\langle n \rangle}$ for large $\langle n \rangle$.

(a) Show that the state at time *t* the joint state takes the form

$$|\Psi(t)\rangle_{AF} = |C(t)\rangle_F \otimes |g\rangle_A + |S(t)\rangle_F \otimes |e\rangle_A$$

where $|C(t)\rangle_F = \sum_{n=0}^{\infty} c_n \cos(\sqrt{n}gt)|n\rangle$, $|S(t)\rangle_F = -i\sum_{n=0}^{\infty} c_{n+1} \sin(\sqrt{n+1}gt)|n\rangle$, $c_n = (\alpha^n / \sqrt{n!})e^{-|\alpha|^2/2}$.

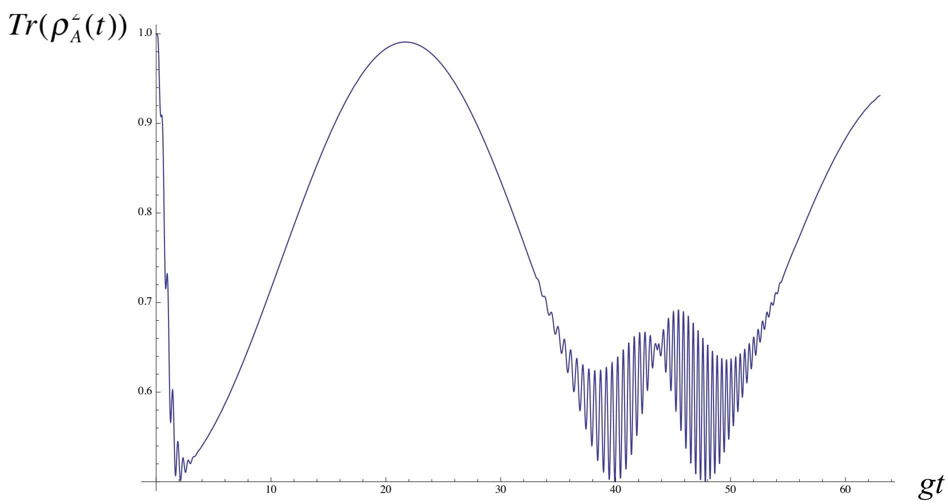
Note $|C(t)\rangle_F, |S(t)\rangle_F$ are not normalized, nor are they orthogonal.

(b) Show that the marginal state of the atom in the $\{|g\rangle, |e\rangle\}$ basis is

$$\hat{\rho}_A(t) = \begin{bmatrix} \langle C(t)|C(t)\rangle & \langle C(t)|S(t)\rangle \\ \langle S(t)|C(t)\rangle & \langle S(t)|S(t)\rangle \end{bmatrix} = \frac{1}{2}\left(\hat{1} + \vec{Q}(t)\cdot\hat{\vec{\sigma}}\right).$$

Write an expression for Bloch vector $\vec{Q}(t)$.

(c) Write the purity of the marginal (a measure of the entanglement between the atom and field), in terms of the Bloch vector. Numerically calculate this and plot as a function of time for $\langle n\rangle = |\alpha|^2 = 49$. Your graph should look like



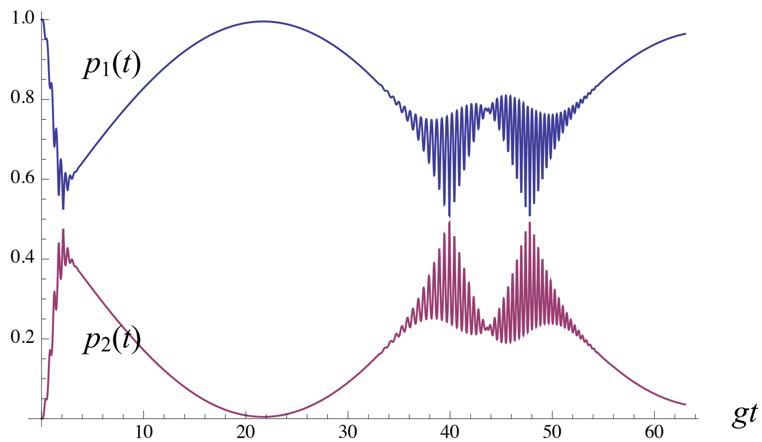
This plots shows a few surprising features. During the collapse the atom and field become highly entangled, as indicated by the rapid degree in the atomic purity. However, at half the revival time, $gt_r/2 \approx \pi\sqrt{\langle n\rangle}$, when the inversion looks to be flat, the purity returns to near unity, indicating that the atom and field become *separable*. The atom and field then become re-entangled. When the Rabi oscillations once again revive, the purity again increases, but nowhere near to unity. Our goal now is to use the Schmidt decomposition to understand this.

(d) Given the initial pure state of the joint system and the unitary evolution according to the JCM, we know that at all times we can express the state in terms of Schmidt decomposition.

$$|\Psi(t)\rangle_{AF} = \sum_{\mu=1}^{2} \sqrt{p_u(t)}\,|u_\mu(t)\rangle_A \otimes |v_\mu(t)\rangle_A .$$

Note, even though the field mode is infinite dimensional, the maximum Schmidt number is 2.

Express the two values of $p_\mu(t)$ in terms of the Bloch vector $\vec{Q}(t)$. Calculate numerically at plot as function of time. Your graphs should look like the following:



Comment on this and what it means for the entanglement.

(e) We can find the Schmidt vectors by the following procedure.

- Find the atomic Schmidt vectors $\left\{ |u_\mu(t)\rangle_A \right\}$ as the eigenvectors of the marginal state $\hat{\rho}_A(t)$ in the standard basis $\left\{ |g\rangle, |e\rangle \right\}$.
- Using $|\Psi(t)\rangle_{AF} = |C(t)\rangle_F \otimes |g\rangle_A + |S(t)\rangle_F \otimes |e\rangle_A = \sum_{\mu=1}^{2} \sqrt{p_u(t)} |u_\mu(t)\rangle_A \otimes |v_\mu(t)\rangle_A$, find an expression for the two Schmidt vectors of the field $\left\{ |v_\mu(t)\rangle_F \right\}$ in terms of $|C(t)\rangle_F, |S(t)\rangle_F, p_u(t)$.

(f) We can see the (approximate) separation between atom and field at half the revival time for large $\langle n \rangle$ as follows. Show that in this limit,

$$g\sqrt{n+1}t_r / 2 \approx g\sqrt{n}t_r / 2 + \pi / 2, \quad c_{n+1} \approx e^{-i\phi} c_n, \text{ where } c_n = (\alpha^n / \sqrt{n!}) e^{-|\alpha|^2/2} \text{ and } \alpha = \sqrt{\langle n \rangle} e^{i\phi} \ .$$

Using this, show that

$$|\Psi(t_r / 2)\rangle_{AF} \approx \left(|g\rangle_A - ie^{-i\phi} |e\rangle_A \right) \otimes |C(t_r / 2)\rangle_F \ .$$

Thus we see that the system is separable, with the atom in an equal superposition depending on the phase of the coherence state.

(g) Extra credit (5 points): More generally show that if $|\Psi(0)\rangle_{AF} = \left(a|g\rangle_A + b|e\rangle_A \right) \otimes |\alpha\rangle_F$

$$|\Psi(t_r / 2)\rangle_{AF} \approx \left(|g\rangle_A - ie^{-i\phi} |e\rangle_A \right) \otimes \left(a|C(t_r / 2)\rangle_F + b|S(t_r / 2)\rangle_F \right)$$

This result shows that *regardless of the atomic initial state*, at half the revival time, the atom goes to the same state. The information about the initial atomic superposition is transferred to the field in a kind of “swap gate.” For large α, the two field states are macroscopically distinguishable. This is kind of “Schrödinger cat”.

Problem 3: Gaussian States in Quantum Optics (EXTRA CREDIT: 35 points)

The set of states whose quadrature fluctuations are Gaussian distributed about a mean value is an important class in quantum optics. These states have Gaussian Wigner functions. In this problem, we explore Gaussian states, their relationship to squeezing, and the canonical algebra of phase space.

Consider a field of *n-modes*, with quadrature defined by an ordered vector:

$$\mathbf{Z} = (X_1, P_1, X_2, P_2, \ldots, X_n, P_n).$$

The operators associated with these quadratures satisfy a set of canonical commutators relations that can be written compactly as,

$$\left[\hat{Z}_i, \hat{Z}_j\right] = \frac{i}{2}\Sigma_{ij}, \text{ where } \Sigma = \bigoplus_{k=1}^{n} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix.}$$

We define an "inner product" in phase space as $\left(\mathbf{Z}\middle|\mathbf{Q}\right) = Z_i \Sigma_{ij} Q_j$ (summed over repeated indices through this problem).

(a) Show that the phase space displacement operator can be written

$$\hat{D}(\mathbf{Z}) = \exp\left\{-i\left(\mathbf{Z}\middle|\hat{\mathbf{Z}}\right)\right\}$$

A *Gaussian state* is one whose Wigner function is a Gaussian function on phase space. Recall the characteristic function of a quantum state is defined $\chi(\mathbf{Z}) = Tr\left(\hat{\rho}\hat{D}(\mathbf{Z})\right)$.

The general form of the characteristic function for a Gaussian state with is:

$$\chi(\mathbf{Z}) = \exp\left\{-\frac{1}{2}\left(\mathbf{Z}\middle|\mathbf{C}\middle|\mathbf{Z}\right) + i\left(\mathbf{d}\middle|\mathbf{Z}\right)\right\}.$$

Where C_{ij} is known as the covariance matrix, and d_i is a real vector.

(b) Show that: $\left\langle \hat{Z}_i \right\rangle = d_i$, and $\frac{1}{2}\left\langle \Delta\hat{Z}_i\Delta\hat{Z}_j + \Delta\hat{Z}_j\Delta\hat{Z}_i \right\rangle = C_{ij}$, where $\Delta\hat{Z}_i \equiv \hat{Z}_i - \left\langle \hat{Z}_i \right\rangle$.

Hint: Recall how moments are found from the characteristic function.

The Gaussian state is thus determined by the mean position in phase space and the covariance of all the fluctuations.

(c) Find the Wigner function for a state with the general form of the characteristic function.

Let us restrict our attention to Gaussian states with zero mean (the mean is irrelevant to the statistics and can always be removed via a displacement operation). Consider now unitary transformations on the state. A particular class of transformations is the set that act as linear canonical transformations, i.e.

$$\hat{U}^{\dagger}\hat{Z}_i\hat{U} = S_{ij}\hat{Z}_j \text{, where } S_{ij} \text{ is a symplectic matrix, defined by } S^T \Sigma S = \Sigma .$$

A unitary map on the state transforms the state according to

$$\chi(\mathbf{Z}) \Rightarrow \chi'(\mathbf{Z}) = Tr\left(\hat{U}\hat{\rho}\hat{U}^{\dagger}\hat{D}(\mathbf{Z})\right) = Tr\left(\hat{\rho}\hat{U}^{\dagger}\hat{D}(\mathbf{Z})\hat{U}\right).$$

(d) Show that for a symplectic transformation, the characteristic function transforms as

$$\chi(\mathbf{Z}) \Rightarrow \chi(\mathbf{SZ})$$

and thus the action of the unitary is to *preserve the Gaussian statistics, by* transforming covariance matrix as $\mathbf{C} \Rightarrow \mathbf{S}^T\mathbf{CS}$.

(e) Show that the following operations preserve Gaussian statistics:

- Linear optics: $\hat{U} = \exp\left(-i\theta_{ij}\hat{a}_i^{\dagger}\hat{a}_j\right)$
- Squeezing: $\hat{U} = \exp\left(\zeta_{ij}^{*}\hat{a}_i\hat{a}_j - \zeta_{ij}\hat{a}_i^{\dagger}\hat{a}_j^{\dagger}\right)$

(f) For each of these, show how the covariance matrix of the Gaussian transforms.

(g) Starting with the vacuum (a Gaussian state) we apply the squeezing operator above. Show that the symplectic transformation on the covariant matrix leads to the expected result.