Physics 581 - Quantum Optics II
Lecture 4: Quasiprobability Functions of the Field

Motivation

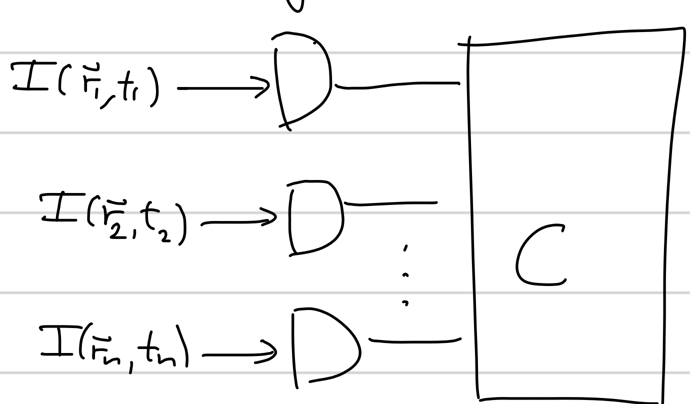
We have defined "classical light" as that whose state can be expressed as a statistical mixture of coherent state

$$\hat{\rho} = \int d^2\{\alpha_k\}\; P(\{\alpha_k\})\; |\{\alpha_k\}\rangle\langle\{\alpha_k\}|$$

$P(\{\alpha_k\})$ = Glauber - Sudharshan "P-representation". Classical $\Leftrightarrow P(\{\alpha_k\}) \geq 0$

This definition of classicality is defined by the condition that a photon counting experiment is describable by the semiclassical theory (i.e. classical wave inducing photo-electric emission).

E.g. Coincidence counting from n-detectors



$$C = \langle :\hat{I}(\vec{r}_1,t_1)\hat{I}(\vec{r}_2,t_2)\cdots\hat{I}(\vec{r}_n,t_n):\rangle \qquad \hat{I}(\vec{r},t) = \hat{E}^{(-)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r},t)$$

normal order

$$= \sum (\;)\; \langle \hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots\hat{a}^\dagger_{k_n}\hat{a}_1\hat{a}_2\cdots\hat{a}_n\rangle$$

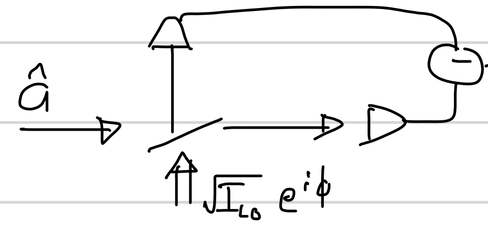
$$\hat{\rho}_{classical} \Rightarrow C = \overline{|E_1|^2|E_2|^2\cdots|E_n|^2}$$

classical statical average

$$= \sum (\quad) \int d^2\{\alpha_k\}\; P(\{\alpha_k\})\; |\alpha_1|^2|\alpha_2|^2\cdots|\alpha_n|^2$$

Any state of the field arising from a classical (maybe noise) current source is described by a statistical mixture of coherent states, so in this sense the light is classical.

We can also, however, consider other detection methods, and ask whether there is a classical, statistical description of the process. Consider, e.g., homodyne detection:



$$\hat{I}_{out} \propto \hat{X}_\phi = \frac{e^{-i\phi}\hat{a} + e^{i\phi}\hat{a}^\dagger}{\sqrt{2}}$$

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right)^2 \right\rangle = \frac{1}{2}\langle \underbrace{\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}_{\text{symmetric order}} \rangle$$

A squeezed state is non-classical in the sense that it cannot be represented as a statistical mixture of coherent states (and thus does not arise as the radiation of a classical noise source), but note

$$\langle \hat{X}^2 \rangle = \int dX \, \frac{e^{-\frac{X^2}{2\Delta X^2}}}{\sqrt{2\pi\Delta X^2}} \, X^2 = \text{classical statistical}$$

average, even when $\Delta X^2 < \frac{1}{2}$ (vacuum limit). So in some sense, the squeezed vacuum can be understood in a classical statistical noise theory.

That is, there exists another way of representing the state in terms of coherent states, such that

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a}+\hat{a}^\dagger}{2}\right)^2 \right\rangle = \int d^2\alpha \, W(\alpha) \left(\frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2$$

$W(\alpha) \equiv$ "Wigner function"

Quantum Mechanics in Phase Space

The fact that there are many different ways to represent a state in terms coherent states follows from the fact that they are "phase space eigenstates."

$$(\hat{X}+i\hat{P})\,|\alpha\rangle = (X+iP)\,|\alpha\rangle \qquad \begin{aligned} X &= \sqrt{2}\,\mathrm{Re}(\alpha) \\ P &= \sqrt{2}\,\mathrm{Im}(\alpha) \end{aligned}$$

$|\alpha\rangle$ is not an eigenstate of a Hermitian operator, and thus, they are not orthogonal, $|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2} \Rightarrow \{|\alpha\rangle\}$ is an <u>over-complete</u> basis: $\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{1}$, but smaller sets will do.

The question of how to represent quantum states in phase space has a long history. From a fundamental perspective, it allows us to compare classical and quantum behavior. Classically, we can formulate Hamiltonian dynamics via Liouville theory.

Classical State: Probability distribution in phase space $W(\vec{q},\vec{p},t)$

Liouville equation for dynamics:

$$\frac{dP}{dt} = -\underbrace{\{H(\vec{q},\vec{p}),\, W(\vec{q},\vec{p},t)\}}_{\text{Poisson bracket}} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial W}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial W}{\partial q_i}\right)$$

$$\overline{f(\vec{q},\vec{p})}\Big|_t = \int d^n\vec{q}\, d^n\vec{p}\; W(\vec{q},\vec{p},t)\, f(\vec{q},\vec{p})$$

expected value of some observable function of the canonical coordinates.

In quantum mechanics we cannot generally represent the state as a classical probability on phase space, because $\hat{X}$ and $\hat{P}$ don't commute. However, use the basis of phase space eigenstates to define quasi probability distributions such that

$$\underbrace{\langle f(\hat{X},\hat{P})\rangle_\sigma}_{} = \int dX\, dP\ \underbrace{W_\sigma(X,P)}_{}\ f_\sigma(X,P)$$

Quantum observable in phase space with ordering σ

Quasi-probability distribution for ordering σ

The issue of operating ordering arises because $\hat{X}$ and $\hat{P}$ don't commute

Consider $f(\hat{X},\hat{P}) = (\hat{X}+\hat{P})^2 = \hat{X}^2 + \hat{P}^2 + \hat{X}\hat{P} + \hat{P}\hat{X}$: Symmetric

$= \hat{X}^2 + \hat{P}^2 + 2\hat{X}\hat{P} + i$: X then P

$= \hat{X}^2 + \hat{P}^2 + 2\hat{P}\hat{X} - i$ P then X

Substituting $\hat{X} \Rightarrow X$, $\hat{P} \Rightarrow P$ gives different functions $f_\sigma(X,P)$.

Often it is convenient to work with complex amplitudes α, α^* (quantum analogues: $\hat{a}, \hat{a}^\dagger$)

Seek $W_\sigma(\alpha,\alpha^*)$ such that $\langle f(\hat{a},\hat{a}^\dagger)\rangle_\sigma = \int d^2\alpha\ W_\sigma(\alpha,\alpha^*)\ f_{-\sigma}(\alpha,\alpha^*)$

Sign to be explained

The function $f_\sigma(\alpha,\alpha^*)$ is defined from power series expansions

Normal order: $f(\hat{a},\hat{a}^\dagger) = \sum C^{+1}_{nm} (\hat{a}^\dagger)^n (\hat{a})^m$

Anti-normal order: $= \sum C^{-1}_{nm} (\hat{a})^m (\hat{a}^\dagger)^n$

Symmetric Order $= \sum C^{0}_{nm} \{(\hat{a}^\dagger)^n (\hat{a})^m\}_{sym}$

$\{(\hat{a}^\dagger)^n(\hat{a})^m\}_{sym} = \binom{m+n}{n}^{-1}$ Permutations of $(\hat{a}^{\dagger n}\hat{a}^m)$

For example: $\{\hat{a}^\dagger\hat{a}^2\}_{sym} = \frac{1}{3}(\hat{a}^\dagger\hat{a}^2 + \hat{a}\hat{a}^\dagger\hat{a} + \hat{a}^2\hat{a}^\dagger)$

Then $f_\sigma(\alpha, \alpha^*) = \sum_{n,m} C_{nm}^{\sigma} (\alpha^*)^n \alpha^m$

Example: $f(\hat{a}^\dagger, \hat{a}) = \hat{n}^2 = \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}$

$$= \underbrace{(\hat{a}^\dagger)^2(\hat{a})^2 + \hat{a}^\dagger \hat{a}}_{C_{22}^{+1} = 1,\ C_{11}^{+1} = 1} = \underbrace{(\hat{a})^2(\hat{a}^\dagger)^2 - 3\hat{a}\hat{a}^\dagger + 1}_{C_{22}^{-1} = 1,\ C_{11}^{-1} = -3,\ C_{00}^{-1} = 1}$$

$$\Rightarrow f_{+1}(\alpha, \alpha^*) = |\alpha|^4 + |\alpha|^2, \quad f_{-1}(\alpha, \alpha^*) = |\alpha|^4 - 3|\alpha|^2 + 1$$

The quasiprobability distribution is the representation of the density operator with respect to a given operator ordering:

$$\hat{\rho} = \sum C_{nm}^{S} \{\hat{a}^{\dagger n} \hat{a}^m\}_{sym} = \sum C_{nm}^{N} \hat{a}^{\dagger n} \hat{a}^m = \sum C_{nm}^{A} \hat{a}^m \hat{a}^{\dagger n}$$

$W(\alpha, \alpha^*) \equiv \sum C_{nm}^{S} \alpha^{*n} \alpha^m \equiv$ Wigner Function
$Q(\alpha, \alpha^*) = \sum C_{nm}^{N} \alpha^{*n} \alpha^m \equiv$ Husimi Function
$P(\alpha, \alpha^*) = \sum C_{nm}^{A} \alpha^{*n} \alpha^m \equiv$ Glauber-Sudharshan P-function

Weyl Group and Operator Ordering

Expressing observables as an order power series is tedious! We can achieve the operator ordering more compactly using the properties of the displacement operator (Cahill & Glauber, Phy Rev 177, 1857 (1969))

$$\hat{D}(\alpha, \alpha^*) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha \hat{a}^\dagger - \alpha^* \hat{a})^n$$

$$= \mathbb{1} + (\alpha \hat{a}^\dagger - \alpha^* \hat{a}) + \frac{1}{2}\left(\alpha^2 \hat{a}^{\dagger 2} - |\alpha|^2(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}) + \alpha^* \hat{a}^2\right) + \cdots$$

$$\Rightarrow \hat{D}(\alpha, \alpha^*) = \underbrace{\sum_{n,m} \frac{\alpha^n (-\alpha^*)^m}{n!\, m!} \left\{(\hat{a}^\dagger)^n (\hat{a})^m\right\}_{sym}}_{\text{Symmetrically ordered power series in } \hat{a} \text{ and } \hat{a}^\dagger}$$

We can use Baker-Campbell-Hausdorff $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ ← when c-number
to express $\hat{D}(\alpha,\alpha^*)$ in normal and antinormal order:

$$\hat{D}(\alpha,\alpha^*) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} = e^{-|\alpha|^2/2} :\hat{D}(\alpha,\alpha^*): \leftarrow \text{normal order}$$

$$\boxed{\Rightarrow :\hat{D}(\alpha,\alpha^*): = \underbrace{\sum_{n,m} \frac{\alpha^n(-\alpha^*)^m}{n!\,m!} (\hat{a}^\dagger)^n (\hat{a})^m}_{\text{normally ordered power series in } \hat{a} \text{ and } \hat{a}^\dagger}}$$

$$\hat{D}(\alpha,\alpha^*) = e^{+|\alpha|^2/2} e^{-\alpha^*\hat{a}} e^{+\alpha\hat{a}^\dagger} = e^{+|\alpha|^2/2} \,\vdots\, \hat{D}(\alpha,\alpha^*) \,\vdots\, \leftarrow \text{anti-normal order}$$

$$\boxed{\Rightarrow \hat{D}(\alpha,\alpha^*) = e^{+\frac{|\alpha|^2}{2}} \underbrace{\sum_{n,m} \frac{\alpha^n(-\alpha^*)^m}{n!\,m!} \hat{a}^m (\hat{a}^\dagger)^n}_{\text{anti-normally ordered power series in } \hat{a} \text{ and } \hat{a}^\dagger}}$$

This construction allows us to define a "continuum" between normally ordered and antinormally ordered power series

$$\hat{D}_\sigma(\alpha,\alpha^*) \equiv e^{\sigma|\alpha|^2/2} \hat{D}(\alpha,\alpha^*)$$

$\hat{D}_0(\alpha,\alpha^*) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ (symmetric order); $\hat{D}_{+1}(\alpha,\alpha^*) = e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}$ (normal order)

$\hat{D}_{-1}(\alpha,\alpha^*) = e^{-\alpha^*\hat{a}} e^{\alpha\hat{a}^\dagger}$ (antinormal order)

Characteristic function

In order to find the ordered power series representation of the density operator, we turn to a tool from classical probability theory, the "characteristic function."

Characteristic function: Fourier transform of the probability distrib.

$$\chi(y) \equiv \int_{-\infty}^{\infty} dx\, e^{-iyx} P(x) = \overline{e^{-iyx}}$$

The characteristic function is used to calculated <u>moments</u> of the distribution:

$$\overline{x^n} = \int_{-\infty}^{\infty} dx\, x^n P(x) = i^n \frac{d^n \chi}{dy^n}\Big|_{y=0}$$

For a joint probability distribution over multiple random variables
$P(x_1, x_2, \dots, x_N) \equiv P(\vec{x})$

$$\boxed{\begin{aligned} &\chi(\vec{y}) \equiv \int d^N x\, e^{-i\vec{y}\cdot\vec{x}} P(\vec{x}), \\ &\Rightarrow \overline{x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}} \equiv \frac{\partial^{n_1}}{\partial(-iy_1)^{n_1}} \frac{\partial^{n_2}}{\partial(-iy_2)^{n_2}} \cdots \frac{\partial^{n_N}}{\partial(-iy_N)^{n_N}} \chi(\vec{y})\Big|_{\vec{y}=0} \end{aligned}}$$

<u>Weyl Representation</u>

We seek the quasiprobability distribution on phase space, with two random variables X, P (for a single mode). We will use the idea of the characteristic function. Note that

$$\hat{D}(\alpha,\alpha^*) = \hat{D}(X,P) = \underbrace{e^{-i(X\hat{P}+P\hat{X})}}_{\text{like plane wave}} = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$$

Define characteristic function of the state of a single mode

$$\boxed{\begin{aligned} \chi(X,P) &= \langle e^{-i(X\hat{P}+P\hat{X})}\rangle = \langle\hat{D}(X,P)\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{D}(X,P)) \\ &= \langle\hat{D}(\alpha,\alpha^*)\rangle = \mathrm{Tr}(\hat{\rho}\,\hat{D}(\alpha,\alpha^*)) = \chi(\alpha,\alpha^*) \end{aligned}}$$

The quasiprobability function is then defined as the inverse Fourier transform

$$W(X,P) = \int \frac{dX' dP'}{2\pi} \chi(X',P') e^{i(X'P-P'X)} = \int \frac{dX'dP'}{2\pi} Tr(\hat{\rho}\hat{D}(X,P)) e^{i(X'P-P'X)}$$

$$W(\alpha) = \int \frac{d^2\beta}{\pi^2} \chi(\beta) e^{\alpha\beta^*-\beta\alpha^*} = \int \frac{d^2\beta}{\pi^2} Tr(\hat{\rho}\hat{D}(\beta)) e^{\alpha\beta^*-\beta\alpha^*}$$

normalization convention

Operator ordering: $\chi(\alpha) = \langle \hat{D}(\alpha) \rangle$ = symmetrically ordered product

General ordering:

$$\boxed{\begin{gathered} W_\sigma(\alpha) = \int \frac{d^2\beta}{\pi^2} \chi_\sigma(\beta) e^{\alpha\beta^*-\alpha^*\beta}, \quad \chi_\sigma(\beta) \equiv Tr(\hat{\rho}\hat{D}_\sigma(\beta)) \\ W_0(\alpha) = W(\alpha): \text{Wigner function}, \quad W_{+1}(\alpha) = Q(\alpha) = \text{Husmi function} \\ W_{-1}(\alpha) = P(\alpha) = \text{Glauber - Sudharshan P- function} \end{gathered}}$$

<u>Weyl Alegbra</u>

To go from the characteristic function to the quasiprobability function we need the equivalent of the Fourier transform in operator space

<u>For states</u>, $\{|X\rangle\}$, $\{|P\rangle\}$ are complete, orthonormal bases.

$$\int_{-\infty}^{\infty} dX\, |X\rangle\langle X| = \int_{-\infty}^{\infty} dP\, |P\rangle\langle P| = \hat{1} \qquad \langle X|X'\rangle = \delta(X-X')$$

$$\langle P|P'\rangle = \delta(P-P')$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dX\, |X\rangle \underbrace{\langle X|\psi\rangle}_{\psi(X)} = \int_{-\infty}^{\infty} dP\, |P\rangle \underbrace{\langle P|\psi\rangle}_{\tilde{\psi}(P)}$$

Plane-wave / Delta function duality:

$$\langle x|x'\rangle = \delta(x-x') = \int \frac{dp}{2\pi}\, e^{-ip(x-x')}$$

$$\delta^{(2)}(\alpha) = \int \frac{d^2\beta}{\pi^2}\, e^{\alpha\beta^* - \alpha^*\beta} \qquad \alpha = \frac{x+ip}{\sqrt{2}},\ \beta = \frac{x'+ip'}{\sqrt{2}}$$

For operators, $\{\hat{D}(\alpha)\}$ is a complete orthonormal set of operators

$$\underbrace{(\hat{D}(\beta)|\hat{D}(\alpha))}_{\text{Inner Product}} = \mathrm{Tr}(\hat{D}^\dagger(\beta)\hat{D}(\alpha)) = \int d^2\gamma \,\langle\gamma|\hat{D}^\dagger(\alpha)\hat{D}(\beta)|\gamma\rangle$$

coherent state

Inner Product $= \pi\,\delta^{(2)}(\alpha-\beta) = \pi\,\delta(x_\alpha - x_\beta)\,\delta(p_\alpha - p_\beta)$

$$\mathrm{Tr}(\hat{D}(\alpha)) = \pi\,\delta^{(2)}(\alpha)$$

Completeness $\displaystyle\int \frac{d^2\alpha}{\pi}\, |\hat{D}(\alpha))(\hat{D}(\alpha)| = \int \frac{d^2\alpha}{\pi}\, |\hat{D}^\dagger(\alpha))(\hat{D}^\dagger(\alpha)| = \mathbb{1}$

$$\Rightarrow |\hat{A}) = \int \frac{d^2\alpha}{\pi}\, |\hat{D}^\dagger(\alpha))(\hat{D}^\dagger(\alpha)|\hat{A})$$

$$\boxed{\hat{A} = \int \frac{d^2\alpha}{\pi}\, \mathrm{Tr}(\hat{D}(\alpha)\hat{A})\,\hat{D}^\dagger(\alpha)}$$ Weyl representation

We can extend the Weyl representation to different operator orderings. Note that

$$\mathrm{Tr}(\hat{A}\hat{D}(\alpha))\hat{D}^\dagger(\alpha) = \mathrm{Tr}(\hat{A}\hat{D}(\alpha)e^{\sigma|\alpha|^2/2})\, e^{-\frac{\sigma|\alpha|^2}{2}}\hat{D}^\dagger(\alpha)$$
$$= \mathrm{Tr}(\hat{A}\hat{D}_\sigma(\alpha))\,\hat{D}^\dagger_{-\sigma}(\alpha)$$

$$\Rightarrow \boxed{\hat{A} = \int \frac{d^2\alpha}{\pi}\, \mathrm{Tr}(\hat{A}\hat{D}_\sigma(\alpha))\,\hat{D}^\dagger_{-\sigma}(\alpha)}$$ Generalized Weyl representation

Finally, we have the delta-function/plane wave duality for operators.

<u>Lemma:</u> (to be proven in homework)

$$\hat{T}_\sigma(\alpha) \equiv \int \frac{d^2\beta}{\pi} \hat{D}_\sigma(\beta) e^{\alpha\beta^* - \alpha^*\beta} = \pi \{ \delta^{(2)}(\alpha^* - \hat{a}^\dagger) \delta^{(2)}(\alpha - \hat{a}) \}_\sigma \text{ (Dual basis)}$$

$$\{\delta(X - \hat{X}) \delta(P - \hat{P})\}_\sigma = \int \frac{dX' dP'}{\pi^2} e^{i(P'\hat{X} - X'\hat{P})} e^{i(X'P - P'X)}$$

(Fourier duality) Note: $\hat{T}_\sigma(\alpha) = \hat{T}^\dagger_\sigma(\alpha)$, $\mathrm{Tr}(\hat{T}_\sigma(\alpha)) = 1$

$\Rightarrow$ The dual basis is also orthonormal and complete

$$\int \frac{d^2\alpha}{\pi} |\hat{T}_{-\sigma}(\alpha))(\hat{T}_{+\sigma}(\alpha)| = \hat{\mathbb{1}} \qquad (\hat{T}_\sigma(\alpha) | \hat{T}_{-\sigma}(\beta)) = \pi \delta^{(2)}(\alpha - \beta)$$

$$\Rightarrow \hat{A} = \int \frac{d^2\alpha}{\pi} \mathrm{Tr}(\hat{T}_{-\sigma}(\alpha) \hat{A}) \hat{T}_{+\sigma}(\alpha)$$

Case of normal order: $\hat{T}_{+1} = \pi \delta(\alpha^* - \hat{a}^\dagger) \delta(\alpha - \hat{a}) = |\alpha\rangle\langle\alpha|$ (See homework)
Thus, as we have seen, projectors on coherent states are not orthogonal. The are orthogonal to the dual $\hat{T}_{-1}(\alpha)$.

<u>Operator-ordered Representation</u>

We have defined operator-ordered power series

$$\hat{A} = \underbrace{\sum_{n,m} c^\sigma_{nm} \{\hat{a}^{\dagger n} \hat{a}^m\}_\sigma}_{\equiv \hat{A}_\sigma(\hat{a}^\dagger, \hat{a})} \Rightarrow$$

σ-order representation

$$A_\sigma(\alpha) = \hat{A}_\sigma(\hat{a} = \alpha, \hat{a}^\dagger = \alpha^*) = \sum_{n,m} c^\sigma_{nm} (\alpha^n)(\alpha^*)^m$$

where $\{\hat{a}^{\dagger n} \hat{a}^m\}_\sigma$ is the operator-ordered product

Normal order: $\sigma = +1$ $\{\hat{a}^{\dagger n} \hat{a}^m\}_{+1} = (\hat{a}^\dagger)^n (\hat{a})^m$

Antinormal order: $\sigma = -1$ $\{\hat{a}^{\dagger n} \hat{a}^m\}_{-1} = \hat{a}^m \hat{a}^{\dagger n}$

Symmetric order: $\sigma = 0$ $\{\hat{a}^{\dagger n} \hat{a}^m\}_0 \rightarrow$ All permutations

$$\hat{D}_\sigma(\alpha) = e^{\frac{\sigma|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \left\langle \begin{array}{ll} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} & (\sigma=+1) \\ e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} & (\sigma=0) \\ e^{-\alpha^*\hat{a}} e^{\alpha\hat{a}^\dagger} & (\sigma=+1) \end{array} \right.$$

$$= \sum \frac{\alpha^n(-\alpha^*)^m}{n!\,m!} \{\hat{a}^{\dagger n}\hat{a}^m\}_\sigma$$

General Weyl-Representation: $\hat{A} = \int \frac{d^2\beta}{\pi} \underbrace{\mathrm{Tr}(\hat{A}\hat{D}_{-\sigma}(\beta))}_{\tilde{A}_{-\sigma}(\beta)} \hat{D}_\sigma^\dagger(\beta)$

$$\Rightarrow \hat{A} = \sum_{n,m} C^\sigma_{n,m} \{(\hat{a}^\dagger)^n(\hat{a})^m\}_\sigma, \qquad C^\sigma_{nm} = \int \frac{d^2\beta}{\pi} \mathrm{Tr}(\hat{A}\hat{D}_{-\sigma}(\beta)) \frac{\beta^n(-\beta^*)^m}{n!\,m!}$$

$$\Rightarrow \boxed{A_\sigma(\alpha) = \int \frac{d^2\beta}{\pi} \tilde{A}_{-\sigma}(\beta)\, \hat{D}_\sigma^\dagger(\beta)\Big|_{\substack{\hat{a}\to\alpha \\ \hat{a}^\dagger\to\alpha^*}} = \int \frac{d^2\beta}{\pi} \tilde{A}_{-\sigma}(\beta)\, e^{\alpha\beta^* - \alpha^*\beta}}$$

For the specific case of the density operator, we define

$$W_\sigma(\alpha) \equiv \frac{1}{\pi} P_\sigma(\alpha), \qquad \tilde{\rho}_\sigma(\beta) \equiv \chi_\sigma(\beta)$$

$$\Rightarrow \boxed{W_\sigma(\alpha) = \int \frac{d^2\beta}{\pi^2} \chi_{-\sigma}(\beta)\, e^{\alpha\beta^* - \alpha^*\beta} : \text{Quasi probability}}$$

Dual basis representation:

$$\boxed{\begin{aligned} A_\sigma(\alpha) &= \int \frac{d^2\beta}{\pi} \mathrm{Tr}(\hat{A}\hat{D}_{-\sigma}(\beta))\, e^{\alpha\beta^* - \alpha^*\beta} = \mathrm{Tr}(\hat{A}\hat{T}_{-\sigma}(\alpha)) \\ &= \mathrm{Tr}(\hat{A}\{\delta(\alpha-\hat{a})\,\delta(\alpha^*-\hat{a}^\dagger)\}_{-\sigma}] \end{aligned}}$$

$$\hat{A} = \int \frac{d^2\alpha}{\pi} \mathrm{Tr}(\hat{A}\hat{T}_{-\sigma}(\alpha))\, \hat{T}_\sigma(\alpha) \Rightarrow \boxed{\hat{A} = \int \frac{d^2\alpha}{\pi} A_\sigma(\alpha)\, \hat{T}_\sigma(\alpha)}$$

$$\boxed{W_\sigma(\alpha) = \frac{1}{\pi} \mathrm{Tr}(\hat{\rho}\,\hat{T}_{-\sigma}(\alpha)) = \frac{1}{\pi} \mathrm{Tr}(\hat{\rho}\,\{\delta(\alpha^*-\hat{a}^\dagger)\,\delta(\alpha-\hat{a})\}_{-\sigma})}$$

Expectation values

Consider $\langle \hat{A} \rangle = Tr(\hat{\rho}\hat{A}) = (\hat{\rho}|\hat{A})$ (operator inner product)

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} Tr(\hat{\rho}\, \hat{T}_\sigma(\alpha))\, \hat{T}_\sigma(\alpha)$$

$$\Rightarrow \hat{A} = \int d^2\alpha\, \frac{1}{\pi} Tr(\hat{\rho}\, \hat{T}_\sigma(\alpha))\, Tr(\hat{A}\, \hat{T}_{-\sigma}(\alpha))$$

$$\Rightarrow \boxed{\langle \hat{A} \rangle = \int d^2\alpha\, W_\sigma(\alpha)\, A_{-\sigma}(\alpha) = \int \frac{d^2\beta}{\pi} \chi_{-\sigma}(\beta)\, \tilde{A}_\sigma(\beta)}$$

This is the major result: Any operator function of $\hat{X}, \hat{P}$ has different representations as functions on phase-space $A_\sigma(X,P)$

$$\langle \{ \hat{a}^{+n} \hat{a}^m \}_{sym} \rangle = \int d^2\alpha\, (\alpha^*)^n \alpha^m\, W(\alpha) \qquad \text{Wigner}$$

$$\langle (\hat{a}^+)^n (\hat{a})^m \rangle = \int d^2\alpha\, (\alpha^*)^n (\alpha)^m\, P(\alpha) \qquad \text{Glauber}$$

$$\langle (\hat{a})^m (\hat{a}^+)^n \rangle = \int d^2\alpha\, (\alpha^*)^n (\alpha^m)\, Q(\alpha) \qquad \text{Husmi}$$

These moments can also be calculated from the characteristic function associated with the given ordering ordering

$$\langle \{ \hat{a}^{+n} \hat{a}^m \}_\sigma \rangle = \frac{\partial^n}{\partial \beta^n} \frac{\partial^m}{(\partial \beta^*)^m} \chi_\sigma \Big|_{\beta=0}$$

This is often the most convenient form

Properties of the quasiprobability functions

Real: $W_\sigma^*(\alpha) = \mathrm{Tr}(\hat\rho\, \hat T_\sigma(\alpha))^* = \mathrm{Tr}(\hat\rho^\dagger \hat T_\sigma^\dagger(\alpha)) = W_\sigma(\alpha)$

Normalization: $\hat\rho = \int d^2\alpha\, W_\sigma(\alpha)\, \hat T_{-\sigma}(\alpha) = \int \frac{d^2\beta}{\pi} \chi_{-\sigma}(\beta)\, \hat D^\dagger_{+\sigma}(\beta)$

$$\mathrm{Tr}(\hat\rho) = \int d^2\alpha\, W_\sigma(\alpha) \underbrace{\mathrm{Tr}(\hat T_{-\sigma}(\alpha))}_{1} = \int \frac{d^2\beta}{\pi} \chi_{-\sigma}(\beta) \underbrace{\mathrm{Tr}(\hat D^\dagger_\sigma(\beta))}_{\pi\,\delta^{(2)}(\beta)}$$

$$\Rightarrow \boxed{\mathrm{Tr}(\hat\rho) = \int d^2\alpha\, W_\sigma(\alpha) = \chi_{-\sigma}(0)}$$

In general, however $W_\sigma(\alpha) \ngeq 0 \Rightarrow$ Not probability distribution

Existence: So far we have not actually proven that and of the operator ordered power series actually converge. To prove this we turn to the Fourier analysis.

The class of function for which the Fourier transform exists are the square normalizable functions

$$\|f\|^2 \equiv \int \frac{d^2\alpha}{\pi} |f(\alpha)|^2 \quad : \text{Finite}$$

If $\|f\|$ is finite $\tilde f(\beta) = \int \frac{d^2\alpha}{\pi} f(\alpha)\, e^{\alpha\beta^* - \alpha^*\beta}$ exists, and

$$\|f\|^2 = \int \frac{d^2\beta}{\pi} |\tilde f(\beta)|^2$$

Bound operators: $\|\hat A\|^2 \equiv (\hat A|\hat A) = \mathrm{Tr}(\hat A^\dagger \hat A) = \int \frac{d^2\alpha}{\pi} \hat A^*_\sigma(\alpha)\, \hat A_{-\sigma}(\alpha)$

$$\|\hat A\|^2 = \int \frac{d^2\alpha}{\pi} |A_0(\alpha)|^2 = \int \frac{d^2\beta}{\pi} |\tilde A(\beta)|^2 : \text{Bounded if finite}$$

Wigner function (W representation)

$\hat{\rho}$ is a bounded operator $\Rightarrow$ $\mathrm{Tr}(\hat{\rho}^2) = \int d^2\beta \, |\chi_0(\beta)|^2 \leq 1$

$\Rightarrow$ The characteristic function of Wigner function is in $\mathcal{L}_2(\mathbb{R}^2)$

$\Rightarrow$ $\boxed{W(\alpha) \text{ always exists for a physical state}}$

More generally the symmetrically ordered Weyl symbol $\hat{A}_0(\alpha)$ always exists

Husmi distribution (Q-representation)

$$Q(\alpha) = W_{+1}(\alpha) = \int \frac{d^2\beta}{\pi} \chi_{-1}(\beta) \, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{-1}(\beta) = \mathrm{Tr}(\hat{\rho} \, \hat{D}_{-1}(\beta)) = e^{-|\beta|^2/2} \chi_0(\beta)$$

$\Rightarrow$ The characteristic function of Q is always square normalizable

$\Rightarrow$ $\boxed{Q(\alpha) \text{ always exists for a physical state}}$

More generally the normally ordered Weyl symbol $\hat{A}_{+1}(\alpha)$ always exists

Glauber distribution (P-representation)

$$P(\alpha) = W_{-1}(\alpha) = \int \frac{d^2\beta}{\pi} \chi_{+1}(\beta) \, e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{+1}(\beta) = \mathrm{Tr}(\hat{\rho} \, \hat{D}_{+1}(\beta)) = e^{+|\beta|^2/2} \chi_0(\beta) : \text{Generally \underline{unbounded}}$$

$\Rightarrow$ $\boxed{P(\alpha) \text{ Only exists if } \chi_0(\beta) \text{ falls of at least as fast as fast as } e^{-|\beta|^2/2}.}$

More generally, the antisymmetrically ordered Weyl symbol $\hat{A}_{-1}(\alpha)$ doesnt exist as a "tempered" function.

Properties of Q, P, W

Husmi Q-representation

$$Q(\alpha) = \frac{1}{\pi}\mathrm{Tr}\left(\hat{\rho}\,\hat{T}_{-1}(\alpha)\right) = \frac{1}{\pi}\mathrm{Tr}\left(\hat{\rho}\,|\alpha\rangle\langle\alpha|\right)$$

$$\Rightarrow \boxed{Q(\alpha) = \frac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi} \geq 0 \Rightarrow \text{the Husimi distribution is true probability distribution}}$$

Note for a pure state $Q(\alpha) = \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2$. This is the most natural way to define a probability distribution on phase space, as Husmi did in 1940

While $Q(\alpha)$ always exists and is always positive, it is rarely usely for making predictions. This is because

$$\langle\hat{A}\rangle = \int d^2\alpha\; Q(\alpha)\, A_{-1}(\alpha)$$ ← antisymmetric Weyl symbol

Because $A_{-1}(\alpha)$ rarely exists, this expression is not useful. That's what we expect, otherwise all of QM in phase space reduces to a classical model.

Glauber-Sudharshan P-representation

$$P(\alpha) = \frac{1}{\pi}\mathrm{Tr}\left(\hat{\rho}\,\hat{T}_{+1}(\alpha)\right) \Rightarrow \hat{\rho} = \int d^2\alpha\; P(\alpha)\,\hat{T}_{-1}(\alpha)$$

$$\Rightarrow \hat{\rho} = \int d^2\alpha\; P(\alpha)\,|\alpha\rangle\langle\alpha| :$$ Statistical mixture of coherent states

$$\langle\hat{A}\rangle = \int d^2\alpha\; P(\alpha)\,\langle\alpha|\hat{A}|\alpha\rangle = \int d^2\alpha\; P(\alpha)\, A_{+1}(\alpha)$$ ← always exists

$\Rightarrow$ $P(\alpha)$ doesn't always exist, but when it does and ≥ 0 the state is essential classical

Wigner-Function W-representation

$$\langle \hat{A} \rangle = \int d^2\alpha \; W(\alpha) A_0(\alpha)$$

The Wigner function always exists, and it always is useful for making predictions. Those predictions are only equivalent to a classical statistical theory if $W(\alpha) \geq 0 \;\; \forall \alpha$.

Other properties: (Proved in homework)

- Wigner's original formula: $W(X,P) = \frac{1}{2} W(\alpha)$ Because $\alpha = \frac{1}{\sqrt{2}}(X+iP)$

$$W(X,P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi} e^{-iPY} \langle X + \tfrac{Y}{2} | \hat{\rho} | X - \tfrac{Y}{2} \rangle = \int_{-\infty}^{\infty} \frac{dY}{2\pi} \psi^*(X+\tfrac{Y}{2}) \psi(X-\tfrac{Y}{2}) e^{-iPY}$$

- Marginals: (probability distribution of one variable in joint distrib)

$$P(X) \equiv \int_{-\infty}^{\infty} dP \; W(X,P) = \langle X | \hat{\rho} | X \rangle \; (= |\langle X | \psi \rangle|^2 \text{ pure state})$$

$$P(P) \equiv \int_{-\infty}^{\infty} dX \; W(X,P) = \langle P | \hat{\rho} | P \rangle \; (= |\langle P | \psi \rangle|^2)$$

Generally, $P(X_\theta) = \int_{-\infty}^{\infty} dP_\theta \; W(X,P) = \langle X_\theta | \hat{\rho} | X_\theta \rangle$ quadrature

Thus, $\langle \Delta \hat{X}_\theta^2 \rangle = \int d^2\alpha \; \Delta X_\theta^2 \; W(\alpha)$

$\Rightarrow$ moments of quadratures are the moments of the Wigner function.

- Hudson's Theorem: $W(\alpha) \geq 0$ iff $W(\alpha)$ is Gaussian and physical only if $\Delta X_\theta^2 \Delta P_\theta^2 \geq \frac{1}{2}$

Relations between P, Q, W

$$W(\alpha) = \frac{2}{\pi}\int P(\beta)\, e^{-2|\beta-\alpha|^2}\, d^2\beta$$

$$Q(\alpha) = \frac{2}{\pi}\int W(\beta)\, e^{-2|\beta-\alpha|^2} d\beta = \frac{1}{\pi}\int P(\beta)\, e^{-|\alpha-\beta|^2}\, d\beta$$

$\Rightarrow$ W is a "smoothed" version of P, Q is a "smoothed" version of W. Q is the most "coarse grained" (least rippled).

Examples

- Coherent State $|\alpha_0\rangle$, $\rho = |\alpha_0\rangle\langle\alpha_0|$

$$\boxed{Q(\alpha) = \frac{1}{\pi}|\langle\alpha_0|\alpha\rangle|^2 = \frac{1}{\pi} e^{-|\alpha-\alpha_0|^2} \Rightarrow \frac{1}{2\pi} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} e^{-\frac{(p-p_0)^2}{2\sigma_p^2}}, \quad \sigma_x^2 = \sigma_p^2 = 1}$$

There are many routes to determining other distributions
We can read off $P(\alpha)$ by eye

$$\hat{\rho} = \int d\alpha\, P(\alpha)|\alpha\rangle\langle\alpha| = |\alpha_0\rangle\langle\alpha_0| \Rightarrow \boxed{P(\alpha) = \delta(\alpha-\alpha_0)}$$

$$W(\alpha) = \frac{2}{\pi}\int \frac{d^2\beta}{\pi}\, P(\beta)\, e^{-2|\beta-\alpha|^2}$$

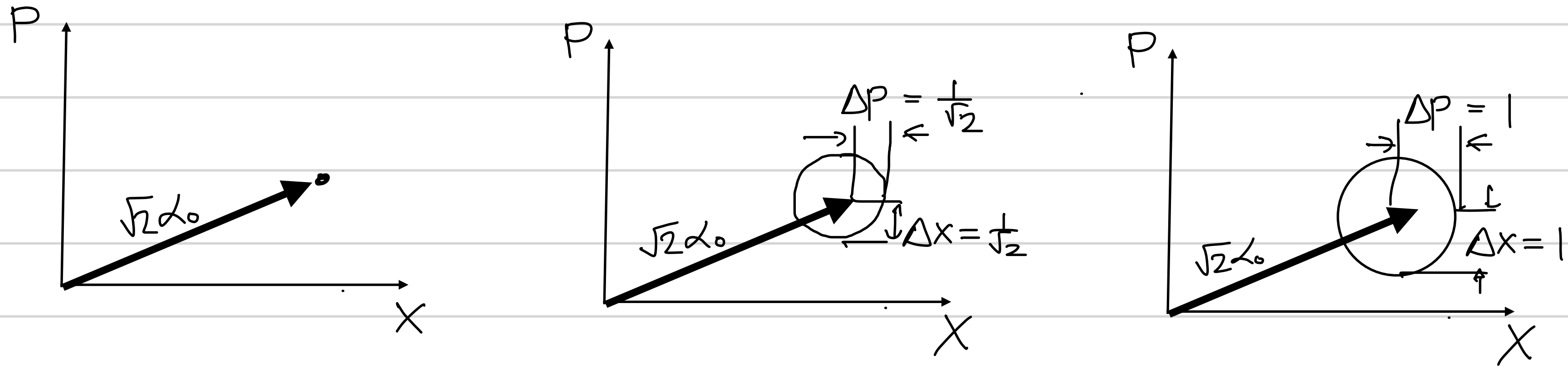
$$\Rightarrow \boxed{W(\alpha) = \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2} \Rightarrow \frac{1}{\sqrt{2\pi\Delta x^2}} e^{-\frac{(x-x_0)^2}{2\Delta x^2}} \frac{1}{\sqrt{\pi\Delta p^2}} e^{-\frac{(p-p_0)^2}{2\Delta p^2}} \quad \Delta x = \Delta p = \frac{1}{\sqrt{2}}}$$

Alternative: Characteristic function

$$\chi_0(\beta) = Tr(\hat{\rho}\,\hat{D}(\beta)) = \langle\alpha_0|\hat{D}(\beta)|\alpha_0\rangle = e^{-|\beta|^2/2}\, e^{\beta\alpha_0^* - \alpha_0^*\beta}$$

$$W_0(\alpha) = \int \frac{d^2\beta}{\pi}\, \chi_0(\beta)\, e^{\alpha\beta^* - \alpha^*\beta^2} \quad \text{Gaussian integral}$$

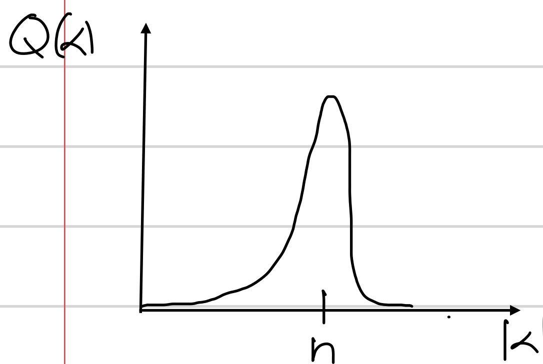
The P, W, Q are Gaussians of different widths

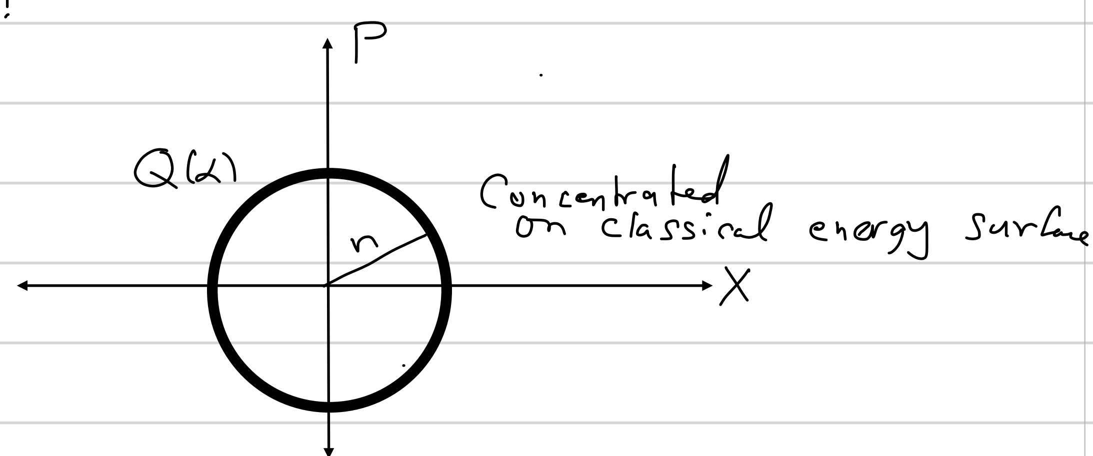


These "uncertainty bubbles" we have been drawing can be interpreted as the contours of the wigner function.

• Fock State

$$Q(\alpha) = \frac{1}{\pi}|\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{(|\alpha|^2)^n}{\pi n!}$$





Symmetric Characteristic function:

$$\chi_0(\beta) = \langle n|\hat{D}(\beta)|n\rangle = e^{-|\beta|^2/2}\mathcal{L}_n(|\beta|^2)$$ ← Laguerre polynomial

$$\Rightarrow W(\alpha) = \int\frac{d^2\beta}{\pi}\chi_0(\beta)e^{\alpha\beta^* - \beta^*\alpha} = \frac{2(-1)^m}{\pi}e^{-2|\alpha|^2}\mathcal{L}_n(4|\alpha|^2)$$