Lecture 9: Master Equation: Examples

Last lecture we introduced the Lindblad form of the master equation, the most general Markov equation consistent with CP-maps:

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{L}_{relax}[\hat{\rho}]$$

$$\mathcal{L}_{relax} = \sum_{\mu}\left[-\frac{1}{2}\left(\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\hat{\rho} + \hat{\rho}\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\right) + \hat{L}_{\mu}\hat{\rho}\hat{L}_{\mu}^{\dagger}\right]$$

The set $\{\hat{L}_{\mu}\}$ are the Lindblad "jump operators" with $\gamma^{\mu}_{j\to j'} = |\langle j'|\hat{L}_{\mu}|j\rangle|^2$ the transition rate from $|j\rangle \to |j'\rangle$ according to a process μ.

Example 1: Two-level atom in a reservoir of black-body radiation

A canonical problem in quantum optics is a two level atom coupled to thermal reservoir of black-body radiation. Including the quantum fluctuations, this also include spontaneous emission in the vacuum (zero-temperature reservoir). The Lindblad equation follow from the usual system+environment Born-Markov approximation, with

$$\hat{H}_{total} = \underbrace{\frac{\hbar\omega_A}{2}\hat{\sigma}_z}_{H_S} + \underbrace{\sum_k \hbar\omega_k a_k^\dagger a_k}_{H_E} + \underbrace{\hbar\sum_k \left(g_k a_k \sigma_+ + g_k^* a_k^\dagger \sigma_-\right)}_{H_{SE}}$$

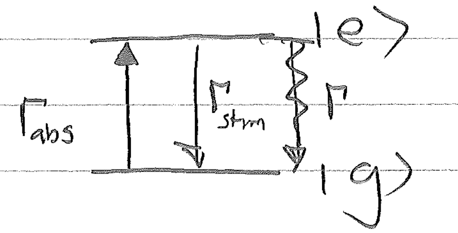
The field is the "environment" in a thermal state

$$\hat{\rho}_E(0) = \prod_k \frac{e^{-\beta\hbar\omega_k a_k^\dagger a_k}}{Z_k} = \prod_k \frac{\bar{n}_k^{\,n_k}}{(\bar{n}_k+1)^{n_k+1}} |n_k\rangle\langle n_k|$$

where $\beta = \frac{1}{k_B T}$, $Z_k = \frac{1}{1-e^{-\beta\hbar\omega_k}}$, $\bar{n}_k = \frac{1}{e^{\beta\hbar\omega_k}-1}$

Note: at $\beta \to \infty$ $(T=0)$ $\hat{\rho}_E(0) \to |vac\rangle\langle vac|$

The interaction of the atom and field leads to absorption + emission



Γ_{abs} = absorption rate

Γ_{stim} = stimulated emission rate

Γ = spontaneous emission rate

According to the Einstein A-B relations

$$\Gamma_{abs} = \Gamma_{stim} = \bar{n}\Gamma$$

$$\text{where } \bar{n} = \bar{n}(\omega_{eg}) = \frac{1}{e^{\beta\hbar\omega_{eg}}-1}$$

We thus have two Lindblad operators defined by

$$\Gamma_{abs} = |\langle e|\hat{L}_{abs}|g\rangle|^2 = \bar{n}\Gamma \Rightarrow \hat{L}_{abs} = \sqrt{\bar{n}\Gamma}\,\hat{\sigma}_+$$

$$\Gamma_{emis} = |\langle g|\hat{L}_{emiss}|e\rangle|^2 = (\bar{n}+1)\Gamma \Rightarrow \hat{L}_{emis} = \sqrt{(\bar{n}+1)\Gamma}\,\hat{\sigma}_-$$

We thus have the Master Eqn for the Atom

$$\frac{d\hat{\rho}_A}{dt} = -\frac{i}{\hbar}[\hat{H}_A, \hat{\rho}_A] + \mathcal{L}_{relax}[\hat{\rho}_A]$$

$$\mathcal{L}_{relax}[\hat{\rho}_A] = -\frac{\Gamma}{2}(\bar{n}+1)\left(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_A + \hat{\rho}_A\hat{\sigma}_+\hat{\sigma}_- - 2\hat{\sigma}_-\hat{\rho}_A\hat{\sigma}_+\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_A + \hat{\rho}_A\hat{\sigma}_+\hat{\sigma}_- - 2\hat{\sigma}_+\hat{\rho}_A\hat{\sigma}_-\right)$$

$$\hat{H}_A = \hbar\omega_{eg}|e\rangle\langle e| = \hbar\omega_{eg}\hat{\sigma}_+\hat{\sigma}_-$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -i\omega_{eg}[|e\rangle\langle e|, \hat{\rho}_A] - \frac{\Gamma}{2}(\bar{n}+1)\left(\{|e\rangle\langle e|, \hat{\rho}\} - 2|g\rangle\langle g|\rho_{ee}\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\underset{\text{anti-commutator}}{\{}|g\rangle\langle g|, \hat{\rho}\} - 2|e\rangle\langle e|\rho_{gg}\right)$$

<u>Evolution of matrix elements</u>

$$\frac{d}{dt}\rho_{ee} = \frac{d}{dt}\langle e|\hat{\rho}|e\rangle = \underbrace{-\Gamma(\bar{n}+1)}_{\text{emission}}\rho_{ee} + \underbrace{\Gamma\bar{n}}_{\text{absorbtion}}\rho_{gg}$$

$$\frac{d}{dt}\rho_{gg} = \frac{d}{dt}\langle g|\hat{\rho}|g\rangle = -\Gamma\bar{n}\rho_{gg} + \Gamma(\bar{n}+1)\rho_{ee}$$

Trace preserving $\frac{d}{dt}(\rho_{gg} + \rho_{ee}) = 0$

Steady State $\Rightarrow$ detailed balance

$$\Rightarrow \frac{d}{dt}\hat{\rho} = 0 \Rightarrow \frac{\rho_{ee}}{\rho_{gg}} = \frac{\bar{n}}{\bar{n}+1} = \frac{(e^{\beta\hbar\omega_{eg}}-1)^{-1}}{(e^{\beta\hbar\omega_{eg}}-1)^{-1}+1}$$

$$= e^{-\beta\hbar\omega_{eg}} \quad \text{Boltzmann!} \checkmark$$

Thus, in steady state the atom come to equilibrium with the bath, as expected

In fact, Einstein <u>derived</u> the spontaneous emission rate to get thermal equilibrium (see: Einstein A/B coefficients)

Decay of coherences (in the absence of coherent driving)

$$\frac{d}{dt} \rho_{eg} = \frac{d}{dt} \langle e | \hat{\rho} | g \rangle$$

$$= -i\omega_{eg} \rho_{eg} - \frac{\Gamma}{2}(2\bar{n}+1) \rho_{eg}$$

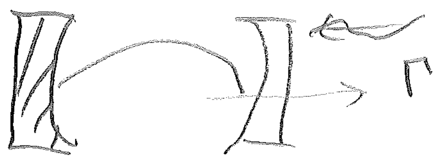
Decay of coherences $\gamma = \dfrac{\Gamma_e + \Gamma_g}{2}$ (in the absence of collisions)

Note: Without coherent drives <u>separate</u> equations of coherences and populations

Another example: Damped SHO

Given oscillator @ freq ω_0 coupled to a bath of <u>thermal oscillators</u>

E.g. mode in leaky cavity



$$\hat{L}_{abs} = \sqrt{\Gamma \bar{n}}\, \hat{a}^\dagger$$
$$\hat{L}_{emis} = \sqrt{\Gamma(\bar{n}+1)}\, \hat{a}$$

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}(\bar{n}+1)\left[\{\hat{a}^\dagger\hat{a},\hat{\rho}\} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right]$$
$$- \frac{\Gamma}{2}\bar{n}\left[\{\hat{a}\hat{a}^\dagger,\hat{\rho}\} - 2\hat{a}^\dagger\hat{\rho}\hat{a}\right]$$

Derived in same Born-Markov approx with

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger\hat{a} + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k$$
$$+ \sum_k \left(g_k \hat{b}_k^\dagger \hat{a} + g_k^* \hat{b}_k \hat{a}^\dagger\right)$$

linear coupling

(In condensed matter literature known as "Caldera-Legget" model)

Consider the evolution of expectation values of observables:

Aside: $\frac{d}{dt}\langle \hat{A}\rangle = \frac{d}{dt}\mathrm{Tr}(\rho \hat{A}) = \mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{A}\right)$

$$\Rightarrow \frac{d}{dt}\langle \hat{A}\rangle = -\frac{1}{2}\sum_\mu \mathrm{Tr}\left(\hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho}\hat{A} + \hat{\rho}\hat{L}_\mu^\dagger \hat{L}_\mu \hat{A} - 2\hat{L}_\mu \rho \hat{L}_\mu^\dagger \hat{A}\right)$$

$$= -\frac{1}{2}\sum_\mu \mathrm{Tr}\left[\left(\hat{A}\hat{L}_\mu^\dagger \hat{L}_\mu + \hat{L}_\mu^\dagger \hat{L}_\mu \hat{A} - 2\hat{L}_\mu^\dagger \hat{A}\hat{L}_\mu\right)\hat{\rho}\right]$$

$$\boxed{\frac{d}{dt}\langle \hat{A}\rangle = -\frac{1}{2}\sum_\mu\left(\langle \hat{L}_\mu^\dagger [\hat{L}_\mu, \hat{A}]\rangle + \langle [\hat{A}, \hat{L}_\mu^\dagger]\hat{L}_\mu\rangle\right)}$$

For example, in damped SHO: mean excitation

$$\frac{d}{dt}\langle \hat{n}\rangle = -\frac{\Gamma}{2}(\bar{n}+1)\left(\langle \hat{a}^\dagger[\hat{a},\hat{n}]\rangle + \langle [\hat{n},a^\dagger]\hat{a}\rangle\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\langle \hat{a}[\hat{a}^\dagger,\hat{n}]\rangle + \langle [\hat{n},\hat{a}]\hat{a}^\dagger\rangle\right)$$

$$= -\frac{\Gamma}{2}(\bar{n}+1)\left(+\langle \hat{a}^\dagger a\rangle + \langle \hat{a}^\dagger a\rangle\right) - \frac{\Gamma}{2}\bar{n}\left(-\langle \hat{a}\hat{a}^\dagger\rangle - \langle \hat{a}\hat{a}^\dagger\rangle\right)$$

$$= -\Gamma(\bar{n}+1)\langle \hat{n}\rangle + \Gamma\bar{n}\left(\langle \bar{n}\rangle + 1\right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle \hat{n}\rangle = -\Gamma\langle \hat{n}\rangle + \Gamma\bar{n}}$$

Solution: $\langle \hat{n}\rangle(t) = \langle \hat{n}\rangle(0) + \bar{n}\left(1-e^{-\Gamma t}\right)$

Steady state: $\boxed{\langle \hat{n}\rangle = \bar{n} \text{ : Thermal equilibrium}}$

Coherences:

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{i}{\hbar}\underbrace{\langle[\hat{a},\hat{H}]\rangle}_{\hbar\omega_0\hat{a}} + \mathrm{Tr}\left(\mathcal{L}_{relax}[\hat{\rho}]\,\hat{a}\right)$$

$$\mathrm{Tr}\left(\mathcal{L}_{relax}[\hat{\rho}]\hat{a}\right) = -\frac{\Gamma}{2}(\bar{n}+1)\left(\langle\hat{a}^{+}[\hat{a},\hat{a}]\rangle + \langle[\hat{a},\hat{a}^{+}]\hat{a}\rangle\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\langle\hat{a}[\hat{a}^{+},\hat{a}]\rangle + \langle[\hat{a},\hat{a}]\hat{a}^{+}\rangle\right)$$

$$= -\frac{\Gamma}{2}(\bar{n}+1)\langle\hat{a}\rangle + \frac{\Gamma}{2}\bar{n}\langle\hat{a}\rangle = -\frac{\Gamma}{2}\langle\hat{a}\rangle$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{a}\rangle = \left(-i\omega_0 - \frac{\Gamma}{2}\right)\langle\hat{a}\rangle}$$ Decay amplitude

$$\Rightarrow \boxed{\langle\hat{a}\rangle(t) = \langle\hat{a}\rangle(0)\, e^{-i\omega_0 t - \frac{\Gamma}{2}t}}$$

<u>Note</u>: The rate of decay is independent of $\langle\hat{a}\rangle$
How is this possible since the decay of ρ_{nn} depends on n?

<u>Look at evolution of coherences of density op</u>

$$\frac{d}{dt}\rho_{n+1,n} = \Big(-i\omega_0 - \big[2\bar{n} + \tfrac{1}{2} + \underset{\substack{\uparrow \\ \text{dependence on } n}}{n}(2\bar{n}+1)\big]\Gamma\Big)\rho_{n+1,n}$$

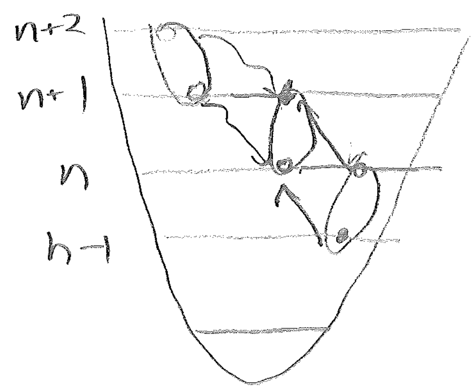
$$+ \sqrt{(n+1)(n+2)}\,(\bar{n}+1)\Gamma\,\rho_{n+2,n+1}$$

$$+ \sqrt{n(n+1)}\,\bar{n}\,\rho_{n,n-1}$$

Note: Coherence decay at rate depending on n

BUT there are also feeding terms

Transfer of coherence



Coherent superposition of $|n+2\rangle$ and $|n+1\rangle$ transfered to superposition of $|n+1\rangle$ and $|n\rangle$

This is only possible because the two decay paths are indistinguishable

This is true only for harmonic ladder

Where the spacing between levels is equal.

Evolution of quadratures

$$\hat{X}_\phi = \frac{\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi}}{\sqrt{2}}$$

$$\Rightarrow \frac{d}{dt}\langle \hat{X}_\phi \rangle = -\Gamma \langle \hat{X}_\phi \rangle \qquad \left(\begin{matrix}\text{in rotating} \\ \text{frame}\end{matrix}\right)$$

$$\frac{d}{dt}\langle \hat{X}_\phi^2 \rangle = -\frac{\Gamma}{2}(\bar{n}+1)\left(\langle \hat{a}^\dagger [\hat{a}, \hat{X}_\phi^2]\rangle + \langle [\hat{X}_\phi^2, \hat{a}^\dagger], \hat{a}\rangle\right)$$
$$-\frac{\Gamma}{2}\bar{n}\left(\langle \hat{a}[\hat{a}^\dagger, \hat{X}_\phi^2]\rangle + \langle [\hat{X}_\phi^2, \hat{a}]\hat{a}^\dagger\rangle\right)$$

Aside: $[\hat{a}, \hat{X}_\phi^2] = \hat{X}_\phi [\hat{a}, \hat{X}_\phi] + [\hat{a}, \hat{X}_\phi]\hat{X}_\phi$
$$= \sqrt{2}\hat{X}_\phi e^{i\phi}$$

$$\Rightarrow \frac{d}{dt}\langle \hat{X}_\phi^2 \rangle = -\frac{\Gamma}{\sqrt{2}}(\bar{n}+1)\langle \hat{a}^\dagger e^{i\phi}\hat{X}_\phi + \hat{X}_\phi \hat{a} e^{-i\phi}\rangle$$
$$-\frac{\Gamma}{\sqrt{2}}\bar{n}\langle -\hat{a}e^{-i\phi}\hat{X}_\phi - \hat{X}_\phi \hat{a}^\dagger e^{i\phi}\rangle$$
$$= -\frac{\Gamma}{\sqrt{2}}(\bar{n}+1)\left[\sqrt{2}\langle \hat{X}_\phi^2\rangle - \frac{1}{\sqrt{2}}\right]$$
$$+\frac{\Gamma}{\sqrt{2}}\bar{n}\left[\sqrt{2}\langle \hat{X}_\phi^2\rangle + \frac{1}{\sqrt{2}}\right]$$
$$= -\Gamma\langle \hat{X}_\phi^2\rangle + \frac{\Gamma}{2}(2\bar{n}+1)$$

$$\Rightarrow \frac{d}{dt}\langle \Delta\hat{X}_\phi^2\rangle = \frac{d}{dt}\left(\langle \hat{X}_\phi^2\rangle - \langle \hat{X}_\phi\rangle^2\right)$$
$$= -\Gamma\langle \Delta\hat{X}_\phi^2\rangle + \frac{\Gamma}{2}(2\bar{n}+1)$$

Solution:

$$\boxed{\langle \Delta \hat{X}_\phi^2 \rangle(t) = \langle \Delta \hat{X}_\phi^2 \rangle(0)\, e^{-\Gamma t} + \left(1 - e^{-\Gamma t}\right)\left(\frac{2\bar{n}+1}{\cancel{\hbar}2}\right)}$$

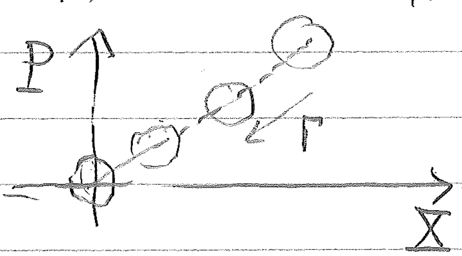
In steady state $\langle \Delta \hat{X}_\phi^2 \rangle = \frac{1}{\cancel{\hbar}2}(2\bar{n}+1)$

Note: Even at zero temperature, the vacuum with $\bar{n}=0$, the quadrature fluctuations damp

$$\langle \Delta \hat{X}_\phi^2 \rangle(t) = \langle \Delta \hat{X}_\phi^2(0) \rangle e^{-\Gamma t} + \frac{1}{2\cancel{\hbar}}\left(1 - e^{-\Gamma t/2}\right)$$

- Example: Coherent State $\langle \Delta \hat{X}_\phi^2 \rangle(0) = \frac{1}{\cancel{\hbar}2}$

$$\Rightarrow \langle \Delta \hat{X}_\phi^2 \rangle(t) = \frac{1}{\cancel{\hbar}2} \quad \forall t$$

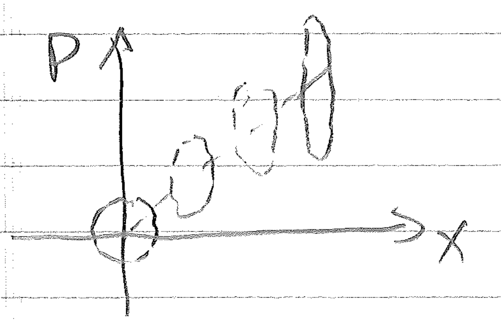


For a coherent state mean amplitude damps but fluctuations unchange

Coherent state is eigenstate of Linblad operator
$\hat{L} = \sqrt{\Gamma}\, \hat{a} \quad \Rightarrow$ "pointer state"

- Example Squeezed State $\langle \Delta \hat{X}^2(0) \rangle = \frac{e^{-2r}}{\cancel{\hbar}2}$

$$\langle \Delta \hat{P}^2(0) \rangle = \frac{e^{+2r}}{2\cancel{\hbar}}$$



$\boxed{\text{Damping kill SQUEEZING}}$

Squeezed state depend on correlated photons.

Fokker – Planck Eq. for quasi-probability fnct

An important tool for dealing with the damped SHO is to consider the quasi-probability distributions, for example the Wigner fnct.

Using characteristic fn $\chi(\beta) = \mathrm{Tr}(\hat{\rho}\,\hat{D}(\beta))$

$$\Rightarrow \frac{\partial \chi(\beta)}{\partial t} = \mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\,\hat{D}(\beta)\right)$$

$$\Rightarrow \frac{\partial W(\alpha)}{\partial t} = \frac{1}{\pi^2}\int d\beta\, \frac{\partial \chi(\beta)}{\partial t}\, e^{\alpha\beta^* - \alpha^*\beta}$$

In terms of the quadratures:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t} W(X,P,t) &= +\frac{\Gamma}{2}\left(\frac{\partial}{\partial X} X + \frac{\partial}{\partial P} P\right) W(X,P,t) \\ &\quad + \frac{\Gamma}{4}(2\bar{n}+1)\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial P^2}\right) W(X,P,t)\end{aligned}}$$

This is known as the Fokker – Plank equation.

- The first term leads to "drift" on the mean phase-space position to the origin, at rate Γ
- The second term leads to "diffusion" of the distribution, spreading in steady state

The F-P equation preserves Gaussians

If W is Gaussian @ $t=0$

then $$W(X,P,t) = \frac{1}{2\pi\,\Delta X(t)\,\Delta P(t)}\, e^{\frac{-X^2}{2\Delta X^2(t)}}\, e^{-\frac{P^2}{2\Delta P^2(t)}}$$

Damping vs. Decoherence

So far our examples have shown that the master equation for the damped SHO have lead to damping and diffusion. These are a "classical" phenomena; familiar in stochastic processes (e.g. Brownian motion). The coupling of a quantum system to an enviroment (reservoir) also leads to very nonclassical effects.

Decoherence of a "Schrödinger Cat"

Suppose we start at $t=0$ with a "macroscopic superposition" of two coherent states

$$|\psi(0)\rangle = \sqrt{\frac{C}{2}}\left(|\alpha_0\rangle + |-\alpha_0\rangle\right) \qquad \text{E.g. } \alpha_0 \text{ real}$$

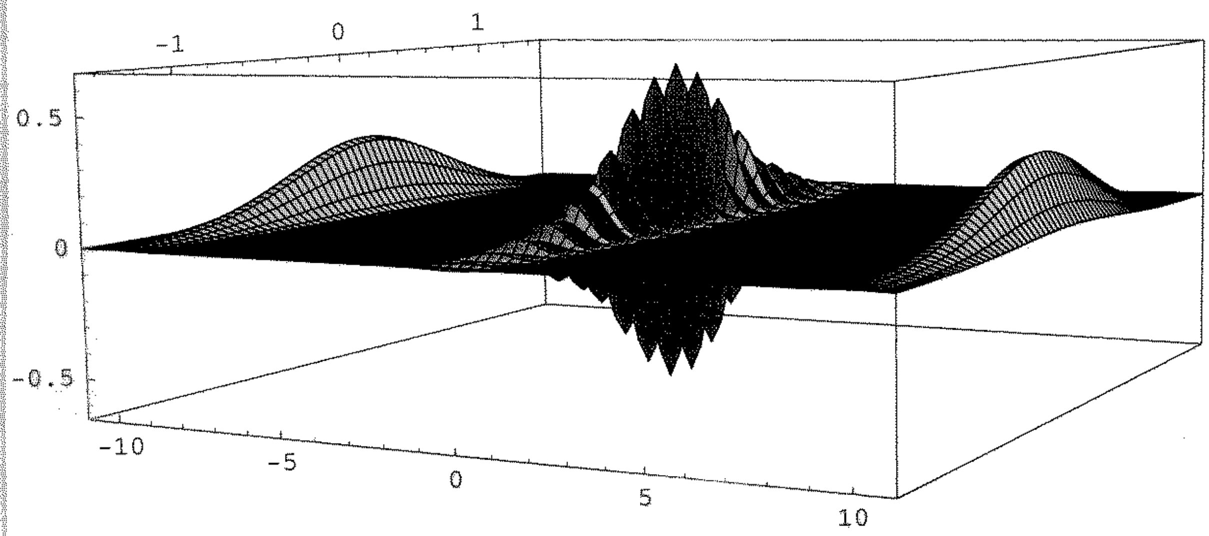
$$\text{where} \quad \alpha_0 = X_0 = \left(\frac{x_0}{\sqrt{\frac{\hbar}{2m\omega_0}}}\right) \quad \text{For mechanical oscillator}$$

$$\text{Macroscopic} \Rightarrow X_0 \gg 1$$

We found the Wigner function in the Problem Set

$$W(X,P,0) = \frac{C}{2}\Big(W_+(xp) + W_-(xp) + \frac{2}{\pi}\cos(X_0\, p)\, e^{-(X^2+P^2)}\Big]$$

$$\text{Where} \quad W_\pm(xp) = \frac{2}{\pi} e^{-\{(x-X_0)^2 + p^2\}}$$



The Wigner function shows localized Gaussians at $x = \pm X_0$, $p = 0$. In addition it Shows rapid oscillations in p at a rate depending on X_0. These come from the "coherence". That is a statistical mixture of coherent states

$$\hat{\rho} = C\left(\frac{1}{2}|\alpha_0\rangle\langle\alpha_0| + \frac{1}{2}|-\alpha_0\rangle\langle-\alpha_0|\right)$$

$$\Rightarrow W(x,p) = C\left(\frac{W_+}{2}(x,p) + \frac{W_-}{2}(x,p)\right)$$

The "interference in phase space" distinguishes the statistical mixture from the "Schrödinger cat" which is in a superposition of two macroscopic possibilities

However, diffusion in the FP equation very quickly wipes out the ~~?~~ coherence for macroscopic superpositions

The oscillating term $\cos(2PX_0)$ under diffusion in P:

$$\Gamma(\bar{n}+\tfrac{1}{2})\frac{\partial^2}{\partial p^2}\cos(2pX_0) = \Gamma(\bar{n}+\tfrac{1}{2})X_0^2\cos(2pX_0)$$

$\Rightarrow$ Amplitude of oscillation decay at rate

$$\gamma_{coh} = X_0^2\,\bar{n}\,\Gamma = \Gamma\left(\frac{x_0^2}{\hbar/m\omega}\right)\bar{n}$$

In high temperature limit $\bar{n} = \frac{kT}{\hbar\omega}$

$$\Rightarrow \gamma_{coh} = \Gamma\left(\frac{X_0^2}{\frac{\hbar^2}{mkT}}\right) = \Gamma\left(\frac{X_0}{\lambda_{DB}}\right)^2$$

where $\lambda_{DB} = \frac{\hbar}{\sqrt{mkT}}$ is the thermal deBroglie wavelength

The lifetime of coherence

$$\tau_{coh} = \frac{1}{\gamma_{coh}} = \tau_{decay}\left(\frac{\lambda_{DB}}{X_0}\right)^2 \qquad \text{where } \tau_{diss} = \frac{1}{\Gamma}$$

For example, at room temperature, $m = 1$ gram, $X_0 = 1$ cm

$$\Rightarrow \frac{\tau_{coh}}{\tau_{decay}} = 10^{-40}$$

Even if $\tau_{decay} = 10^{17}$ s (age of universe)

$\tau_{coh} \sim 10^{-23}$ s