Lecture 13: Different Unravellings of the Master Eqn.

We have seen that the (nonunitary) evolution of the density operator under a "Lindblad" master equation can be interpreted as an average over "quantum trajectories" – stochastically evolving wave functions. This can be thought of as the environment performing a measurement of the system, but not telling us the result. This ~~was~~ evolution generally turns pure states into mixed states

$$|\psi(0)\rangle \longrightarrow \begin{array}{c} |\psi_1(t)\rangle \\ |\psi_2(t)\rangle \\ \vdots \\ |\psi_N(t)\rangle \end{array} \Longrightarrow \hat{\rho}(t) = \lim_{N\to\infty} \sum_i \frac{|\psi_i(t)\rangle\langle\psi_i(t)|}{N}$$

This is the evolution had the environment performed such a measurement, but of course, there is ~~not~~ no such real measurement. In general we can imagine a different measurement made on the environment. This will lead to a different set of quantum trajectories, though the average must lead to the same density since this was an imagined alternative measurement!

These alternative sets of quantum trajectories are known as different unravellings of the master equation. It will be our goal to understand how these are related and how they fit into our modern ~~kn~~ knowledge of quantum measurement theory.

In this lecture I will begin with some basic examples to motivate how different unravellings give different pictures of quantum trajectories.

Formal theory

The Lindblad differential evolution is the CP map

$$\hat{\rho}(t+dt) = \mathcal{L}_{dt}[\hat{\rho}(t)] = \sum_{\mu} \hat{M}_{\mu}(dt)\hat{\rho}(t)\hat{M}_{\mu}^{\dagger}(dt)$$

where Kraus ops are

$$\begin{cases} \hat{M}_0(dt) = \hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}_{eff}\, dt & \mu = 0 \\ \hat{M}_{\mu}(dt) = \sqrt{dt}\, \hat{L}_{\mu} & \mu > 0 \\ \text{with} \quad \hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma}{2}\sum_{\mu} \hat{L}_{\mu}^{\dagger}\hat{L}_{\mu} & \end{cases}$$

From the theory of C-P maps we know that the particular choice of Kraus operators is not unique because the ensemble decomposition of $\hat{\rho}$ into a statistical mixture of pure states is not unique. Indeed

$$\mathcal{L}_{dt}[\hat{\rho}(t)] = \sum_{\mu} \hat{M}_{\mu}(dt)\,\hat{\rho}(t)\,\hat{M}_{\mu}^{\dagger}(dt) = \sum_{\nu} \hat{N}_{\nu}(dt)\,\hat{\rho}(t)\,\hat{N}_{\nu}^{\dagger}(dt)$$

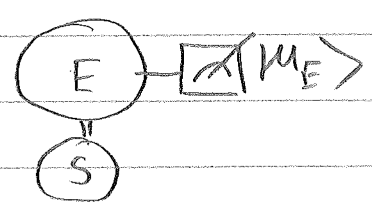
iff $\hat{N}_{\nu}(dt) = \sum_{\mu} \hat{M}_{\mu}(dt)\, u_{\mu\nu}$

where $u_{\mu\nu}$ is a (possibly rectangular) matrix whose rows and columns are orthogonal.

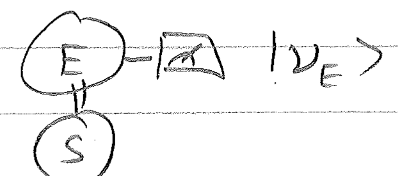
Recall $\hat{M}_{\mu} = \langle \mu_E | \hat{U}_{SE}(dt) | 0_E \rangle$

$\langle \mu_E |$: a basis state for E; $\hat{U}_{SE}$: system-environment coupling; $| 0_E \rangle$: fiducial state of E

The Kraus operator $\hat{M}_\mu$ captures the effect of backaction on the state of system when we measure the environment to be in the state $|\mu_E\rangle$



A different set of Kraus operators corresponds to different measurements on the environment.



$$\hat{N}_\nu = \langle \nu_E | \hat{U}_{SE} | 0_E \rangle$$

The two different sets $\{\hat{M}_\mu\}$ and $\{\hat{N}_\nu\}$ correspond to two different "unravellings" of the density operator evolution into an ensemble of quantum trajectories. The nature of the statistics of these quantum trajectories can be quite different, but the average of $N \to \infty$ # of trajectories must yield the same master equation evolution.

Note: If $\{|\mu_E\rangle\}$ and $\{|\nu_E\rangle\}$ are both orthonormal basis, and have $|0_E\rangle$ in common then

$$\hat{N}_\nu = \sum_\mu{}' u_{\nu\mu} \hat{M}_\mu$$ (Next Page)

where $\mathcal{U}_{\nu\mu}$ is the square unitary matrix

$$\mathcal{U}_{\nu\mu} = \langle \nu_E | \mu_E \rangle .$$

In this case, with $|0_E\rangle$ a common vector

$$\hat{N}_0 = \hat{M}_0 = (\mathbb{1} - \tfrac{i}{\hbar} \hat{H}_{eff}) dt$$

$$\text{so} \quad \sum_{\mu>0}' \hat{M}_\mu^\dagger \hat{M}_\mu = \sum_{\nu>0}' \hat{N}_\nu^\dagger \hat{N}_\nu$$

$$\text{And} \quad \hat{N}_\nu \equiv \sqrt{dt}\, \hat{J}_\nu = \sum_{\mu>0}' u_{\nu\mu} \hat{M}_\mu = \sqrt{dt} \sum_{\mu>0}' u_{\nu\mu} \hat{L}_\mu$$

So the different unravelings correspond to a different set of Lindblad Jump Operators

$$\hat{J}_\nu = \sum_\mu' u_{\nu\mu} \hat{L}_\mu$$

Thus for this case

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [\hat{H}_{eff}, \hat{\rho}]' + \sum_\mu' \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$$

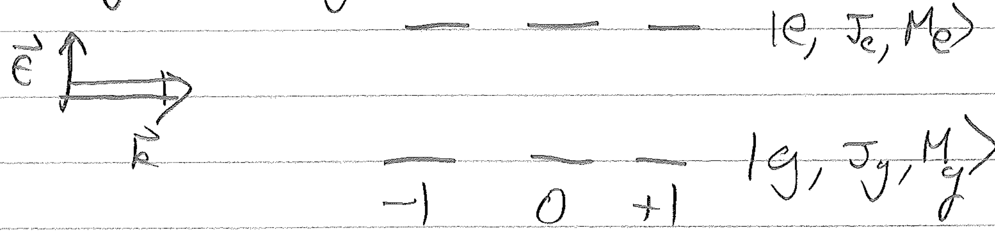
$$= -\frac{i}{\hbar} [\hat{H}_{eff}, \hat{\rho}]' + \sum_\nu' \hat{J}_\nu \hat{\rho} \hat{J}_\nu^\dagger$$

are the same master equation.

the sets $\{\hat{L}_\mu\}$ and $\{\hat{J}_\nu\}$ lead to different unravellings into quantum trajectories.

Example: Coherent Population Trapping

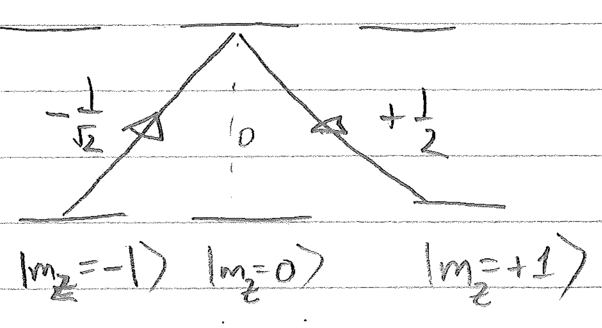
In the previous lecture we studied the decay of atom to a dark state. Specifically we considered an atom with both excited and ground levels have total angular moment $J=1$. These levels are 3-fold degenerate, denoted by quantum numbers $|J,M\rangle$, where M is the projection of angular momentum along a space-fixed quantization axis. The atom is driven by linearly polarized light on resonance.



Last lecture we chose a coordinate system where $\vec{\epsilon} = \vec{e}_y$ and the atoms angular momentum was described relative to $\vec{k} \equiv \hat{e}_z$

W.r.t $\vec{k}$, light has only $\sigma_\pm$ ($\pm$ helicity)

$$\vec{e}_y = \frac{\vec{e}_+ + \vec{e}_-}{i\sqrt{2}}$$



For this field, the dark state is

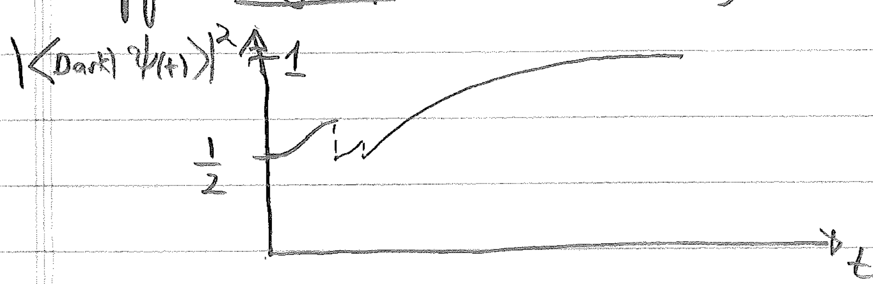
$$|\text{Dark}\rangle = \frac{1}{\sqrt{2}}\left(|g, m_z = 1\rangle + |g, m_z = -1\rangle\right)$$

The atom evolves according to

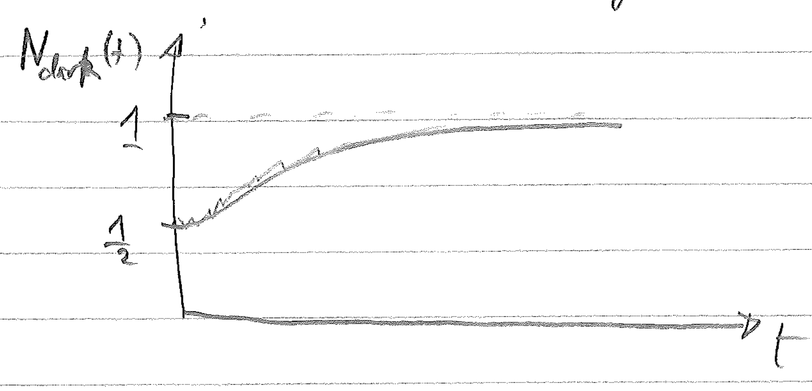
$$\frac{d\rho}{dt} = -\frac{i}{\hbar}\left[\hat{H}_{eff}, \hat{\rho}\right]' + \Gamma \sum_q \hat{D}_q \hat{\rho} \hat{D}_q^{+}$$

where $D_q^{+} = \sum_{M_g, M_e} \underbrace{\langle 1\, M_e | 1\, q\, 1\, M_g\rangle}_{\text{Clebsch-Gordan coeff.}} |e\, M_e\rangle\langle g\, M_g|$

In this "unravelling", if the atom starts in $|\psi(0)\rangle = |g, m_z = -1\rangle$ it arrives in the dark state only through the continuous $\hat{H}_{eff}$ <u>no-jump</u> evolution, With a sample trajectory



And ensemble average over many trajectories



Now we can consider a different unravelling. First, let us emphasize that the operators D_q are define w.r.t. the quantization axis along z

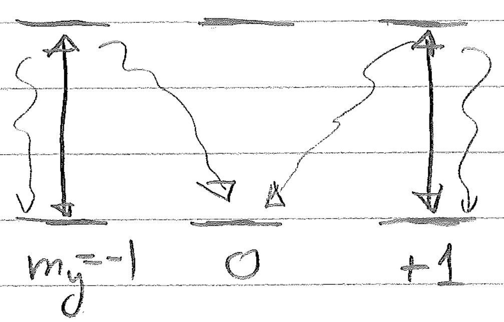
$$\hat{D}_q \equiv \hat{D}_q^{(z)} \qquad \vec{e}_0 \equiv \vec{e}_z, \quad \vec{e}_{\pm 1} \equiv \mp\left(\frac{\vec{e}_x \pm i\vec{e}_y}{\sqrt{2}}\right)$$

We can define a new set of jump-operators where the quantization axis of the atom is along y

$$\hat{D}_q^{(y)} = \sum_{q'=-1}^{1} \underbrace{\mathcal{D}^{(1)}_{qq'}}_{\text{unitary rotation matrix in spherical basis}} \hat{D}_{q'}^{(z)}$$

Now, $\vec{e}_0 = \vec{e}_y \qquad \vec{e}_{\pm 1} = \mp\left(\frac{\vec{e}_z \pm i\vec{e}_x}{\sqrt{2}}\right)$

For this case, the laser photon only drive "π- transitions"



Solid = laser transition
Squiggly = spontaneous photons

The dark state is $|g, m_y=0\rangle$ since the transition $|g, m_y=0\rangle \leftrightarrow |e, m_y=0\rangle$ is forbidden.

Note
$$|J=1, m_y=1\rangle = \frac{1}{2}\left(|J=1, m_z=1\rangle + i\sqrt{2}\,|J=1, m_z=0\rangle - |J=1, m_z=-1\rangle\right)$$
$$|J=1, m_y=0\rangle = \frac{1}{\sqrt{2}}\left(|J=1, m_z=1\rangle + |J=1, m_z=-1\rangle\right)$$
$$|J=1, m_y=-1\rangle = \frac{1}{2}\left(|J=1, m_z=1\rangle - i\sqrt{2}\,|J=1, m_z=0\rangle - |J=1, m_z=-1\rangle\right)$$

Thus, as must be the case, <u>the dark state is always the same</u>, just having different representations in different bases.

$$|\psi_{dark}\rangle = |g, J=1, m_y=0\rangle = \frac{1}{\sqrt{2}}\left(|g, J=1, m_z=+1\rangle + |g, J=1, m_z=-1\rangle\right)$$

However, the way in which we reach the dark-state according to this unravelling is very different because the information we gain about the atomic state is different depending on the photon measurement we do.

Note: $|\tilde{\psi}(0)\rangle = |g, J=1, m_z=-1\rangle$

$$= \frac{1}{\sqrt{2}}|J=1, m_y=0\rangle - \frac{1}{2}\left(|J=1, m_y=1\rangle + |J=1, m_y=-1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|\psi_{dark}\rangle - \frac{1}{\sqrt{2}}|\psi_{bright}\rangle \quad \text{as before}$$

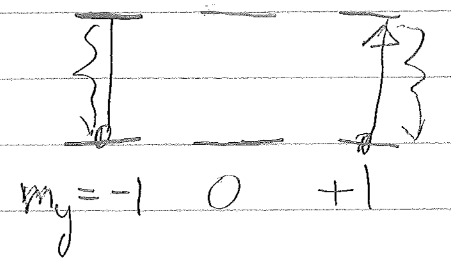
However evolution to the dark state can now take two path

(i) Continuous evolution 50%

(ii) With other 50% optical pumping (jump)

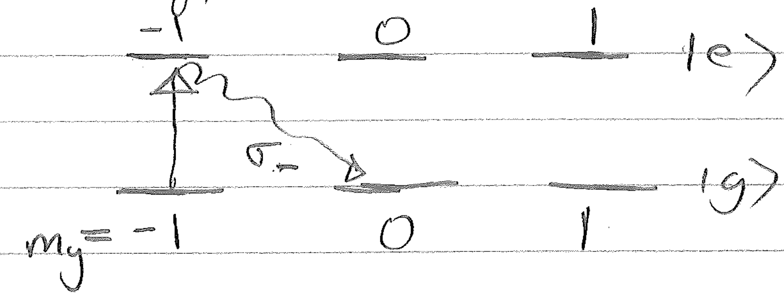
To see how (ii) proceed consider the following possible trajectory. For some time the state "rotates" continuously towards the dark state. Then at some time t a spontaneous photon is emitted. Since this happens from the bright state the atom was in $m_y = \pm 1$.

One possibility is the emission of $q=0$ (π-photon)

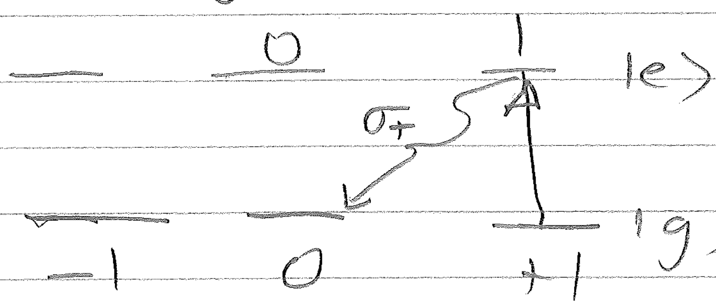


emission of "π-photon"

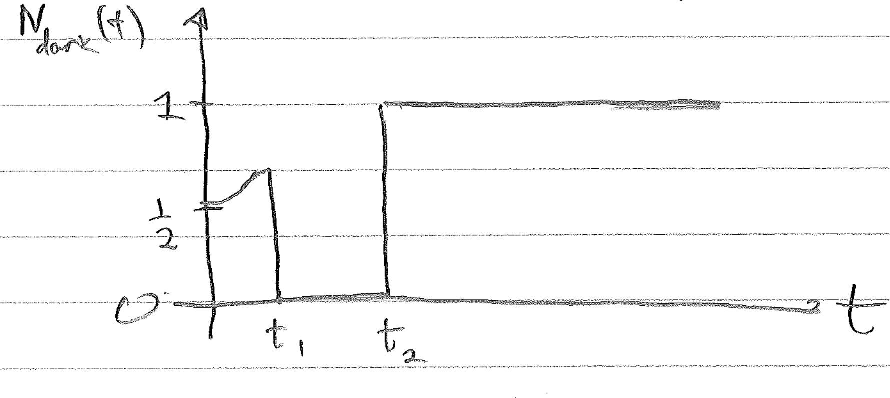
If this happens, then immediately after the jump, the system is totally in the $m_J = \pm 1$ subspace $\Rightarrow$ Only bright state. Over time the atom can emitted more π-spontaneous photons. Eventually, the atom will emit a $\sigma_\pm$ photon



or



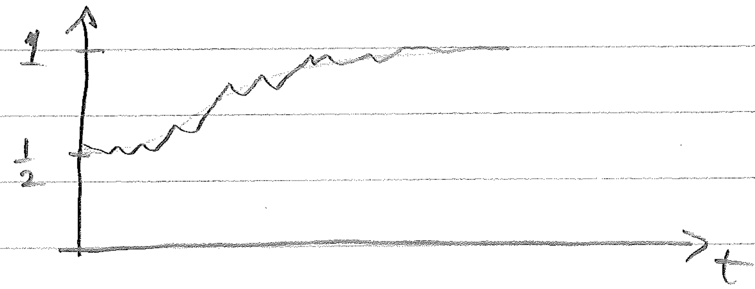
After the jump the atom is in the dark state and evolution stops! A sample trajectory



Note: - At t_1 the atom emitted a π-photon (i.e. $\vec{e}_y$ photon) $\Rightarrow$ atom projected to bright

- At t_2 the atom emitted a σ-photon $\Rightarrow$ atom projected to dark state

Again, averaged over many trajectories, we see the same continuous evolution to the dark state



Note: With this unravelling the finite sum average over N trajectories is much noisier than seen in the unravelling with $\hat{D}^{(z)}$. Thus, though the two unravellings show the same average in the $N \to \infty$ limit, for finite N, one unravelling many be numerically advantageous over the other.