Physics 581: Quantum Optics II
Problem Set #1
Due Tuesday Feb. 4, 2014

Problem 1: Boson Algebra (25 points)
This problem is to give you some practice manipulating the boson algebra. A great source is the classic "Quantum Statistical Properties of Radiation", by W. H. Louisell, reprinted by "Wiley Classics Library", ISBN 0-471-52365-8.

(a) Gaussian integrals in phase-space are used all the time. Show that

$$\int \frac{d^2\beta}{\pi} e^{-A|\beta|^2} e^{\alpha\beta^* - \beta\alpha^*} = \frac{1}{A} e^{-|\alpha|^2/A}.$$

(b) Prove the completeness integral for coherent states

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1} \text{ (Hint: Expand in number states).}$$

(c) The "quadrature" operators in optics are the analogs of X and P, $\hat{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}}$. Show

$$\begin{aligned} \hat{U}^\dagger(\theta)\hat{X}\hat{U}(\theta) &= \cos\theta\, \hat{X} + \sin\theta\, \hat{P} \\ \hat{U}^\dagger(\theta)\hat{P}\hat{U}(\theta) &= \cos\theta\, \hat{P} - \sin\theta\, \hat{X} \end{aligned}, \quad \text{where} \quad \hat{U}(\theta) = e^{-i\theta\hat{a}^\dagger\hat{a}}.$$

Interpret in phase space.

(d) Prove the group property of the displacement operator

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp\{i\,\mathrm{Im}(\alpha\beta^*)\}$$

(e) Show that the displacement operators has the following matrix elements

Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Coherent states: $\langle \alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-|\alpha+\alpha_2-\alpha_1|^2/2} e^{i\,\mathrm{Im}\left(\alpha\alpha_2^* - \alpha_1\alpha^* - \alpha_1\alpha_2^*\right)}$

Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\mathsf{L}_n\left(|\alpha|^2\right)$, where L_n is the Laguerre polynomial of order n

Problem 2: Thermal Light (20 points)

Consider a single mode field in thermal equilibrium at temperature T, Boltzmann factor $\beta = 1/k_B T$. The state of the field is described by the "canonical ensemble",

$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$, $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$ is the Hamiltonian and $Z = Tr\left(e^{-\beta \hat{H}}\right)$ is the partition function.

(a) Remind yourself of the basic properties by deriving the following:

- $\langle n \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$ (the Planck spectrum)
- $P_n = \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}}$ (the Bose-Einstein distribution).

(b) Make a list-plot of P_n for both the thermal state and the coherent state on the same graph as a function of *n,* for each of the following: $\langle n \rangle = 0.1, 1, 10, 100$.

(c) Using the number-state representation, show that

- $\Delta n^2 = \langle n \rangle + \langle n \rangle^2$. How does this compare to a coherent state?
- $\langle \hat{a} \rangle = 0 \Rightarrow \langle \vec{E} \rangle = 0$. How does this compare to a coherent state?

(d) Show that the Glauber-Sudharshan distribution of this state, $P(\alpha) = \frac{1}{\pi \langle n \rangle} e^{-|\alpha|^2/\langle n \rangle}$,

satisfies $\int d^2\alpha \, P(\alpha) |\alpha\rangle\langle\alpha| = \sum_n \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}} |n\rangle\langle n|$. Sketch $P(\alpha)$ in the phase plane.

Problem 3: Twin beams and two-mode squeezed states. (15 points)

Considering the Hamiltonian

$$\hat{H} = i\hbar G\left(\hat{a}_+^\dagger \hat{a}_-^\dagger e^{-i\phi} - \hat{a}_+ \hat{a}_- e^{i\phi}\right),$$

where $\hat{a}_\pm$ are annihilation operators for two modes with frequencies $\omega_\pm$. We will see in class how this arises in nonlinear optics through the process of parametric downconversion. This leads to correlated twin "signal" and "idler" beams as long as the phase matching conditions are satisfied,

$$\omega_p = \omega_s + \omega_i, \ \mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i.$$

Here G is the coupling constant depending on the nonlinearity, pump amplitude, and vacuum mode strength. The state produced is known as a "two-mode squeezed vacuum state", $\hat{S}_{\pm}(\xi)|0\rangle_+ \otimes |0\rangle_- = \exp\left[\xi \hat{a}_+ \hat{a}_- - \xi^* \hat{a}_+^\dagger \hat{a}_-^\dagger\right]|0\rangle_+ \otimes |0\rangle_-$, where $\xi = re^{i\phi}$ is the complex squeezing parameter for an interaction time *t*, $r = Gt$.

(a) Show that the generalized Bogoliubov transformations is

$$\hat{S}_{\pm}^\dagger(\xi)\hat{a}_{\pm}\hat{S}_{\pm}(\xi) = \cosh(r)\hat{a}_{\pm} - e^{-i\phi}\sinh(r)\hat{a}_{\mp}^\dagger.$$

(b) Show that the individual modes, $\hat{a}_{\pm}$, show no squeezing, but that squeezing exists in the *correlation* between the modes. Hint: consider quadratures,

$$\hat{X}_{\pm}(\theta) \equiv \frac{\hat{a}_{\pm}e^{i\theta} + \hat{a}_{\pm}^\dagger e^{-i\theta}}{2} \text{ and then } \hat{Y}(\theta,\theta') \equiv \left(\hat{X}_+(\theta) - \hat{X}_-(\theta')\right)/\sqrt{2}.$$

For the remaining parts, take ξ real.

(c) The two-mode squeezed state is an entangled state between the signal and idler as we know from the perturbative limit of twin photons. Show that in the Fock basis

$$\hat{S}_{\pm}(r)|0\rangle_+ \otimes |0\rangle_- = (\cosh(r))^{-1}\sum_{n=0}^{\infty}(\tanh(r))^n |n\rangle_+ \otimes |n\rangle_-.$$

Hint: Use the "disentangling theorem" (D. R. Traux, Phys. Rev. D **31**, 1988 (1985)):

$$e^{r\left(\hat{a}_+^\dagger \hat{a}_-^\dagger - \hat{a}_+\hat{a}_-\right)} = e^{\Gamma \hat{a}_+^\dagger \hat{a}_-^\dagger} e^{-g\left(\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_- + 1\right)} e^{-\Gamma \hat{a}_+ \hat{a}_-}.$$

$$\text{where } \Gamma = \tanh(r), \ g = \ln(\cosh(r))$$

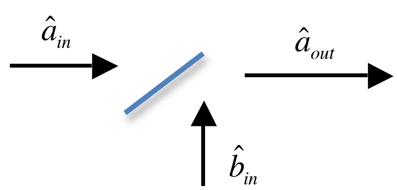
The photons are produced with perfect correlations between the modes. This is known as "number squeezing" in "twin beams.

Problem 4: Shot-noise and vacuum fluctuations (25 Points)

Squeezed states are not robust because of photon absorption (loss). There are many ways to understand this. Squeezing is associated with photon pair-correlations. Loss will randomly remove a photon, and not both in the correlated pair. The remaining photon will then just add shot noise. Another way to understand this is from the perspective of the continuous variables.

Classically, linear loss results in attenuation of the field amplitude $E \rightarrow e^{-\kappa} E = \eta E$, where η is the loss coefficient. We can model this attenuation by a partial transmitting beam splitter, with transmission amplitude η. Quantumly, we cannot make the transformation, $\hat{a} \rightarrow \eta \hat{a}$, because the commutation relations are not preserved. Stated in another way, *we cannot attenuate the vacuum fluctuations.*

(a) Consider a linear transformation between two modes at a beam splitter with transmission amplitude η

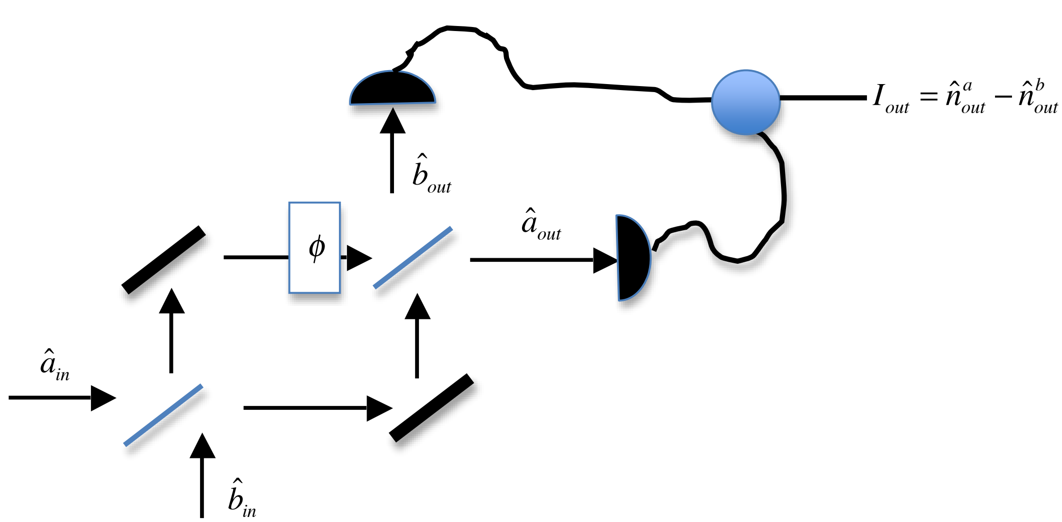


Show that if the vacuum enters in mode-*b*, the output quadrature fluctuations in mode-*a* are

$$\Delta X^2_{\Theta,out} = |\eta|^2 \, \Delta X^2_{\Theta,in} + \left(1 - |\eta|^2\right)\frac{1}{2}.$$

Interpret this result and show how squeezing is degraded by photon loss.

(b) Consider now a Mach-Zender interferometer



Show that output photocurrent operator corresponding to the difference of the photocounts at the two output ports is

$$\hat{I}_{out} = \cos\phi\left(\hat{n}^{a}_{in} - \hat{n}^{b}_{in}\right) - \sin\phi\left(\hat{a}^{\dagger}_{in}\hat{b}_{in} + \hat{b}^{\dagger}_{in}\hat{a}_{in}\right).$$

(c) If the input signal enters through port-*a* and vacuum in port-*b*, then the show the mean output signal and fluctuations are

$$\left\langle \hat{I}_{out} \right\rangle = \cos\phi\left\langle \hat{n}^{a}_{in} \right\rangle,\ \left\langle \Delta\hat{I}^{2}_{out} \right\rangle = \cos^2\phi\left\langle (\Delta\hat{n}^{a}_{in})^2 \right\rangle + \sin^2\phi\left\langle \hat{n}^{a}_{in} \right\rangle$$

(d) The mean output signal is what we know as the interference "fringes." The fluctuations determine the noise. The term $\cos\phi\left\langle \Delta\hat{n}^{a\,2}_{in} \right\rangle$ in $\left\langle \Delta\hat{I}^{2}_{out} \right\rangle$ is the contribution of the noise from the input signal. Show that the term $\sin\phi\left\langle \hat{n}^{a}_{in} \right\rangle$ arises solely do to the vacuum fluctuations entering from the unused port-*b*.

(e) In an application such as LIGO, which measures a tiny effect such as a gravity wave, we typically operate near a node of the fringe so that we are measuring an effect away from zero. Thus we set $\phi = \pi/2 + \delta\phi$, where $\delta\phi << 1$ depends on the strength of the gravity wave. In that case, show that the signal-to-noise ratio (SNR) is

$$SNR = \frac{\left\langle \hat{I}_{out} \right\rangle}{\sqrt{\left\langle \Delta\hat{I}^{2}_{out} \right\rangle}} \approx \delta\phi\sqrt{\left\langle \hat{n}^{a}_{in} \right\rangle}\ .$$

This is known as the "standard quantum limit" and is limited solely by the vacuum noise vacuum noise entering port-*b*.

(f) The seminal work on C. M. Caves, Phys. Rev. D **23**, 1693 (1981), showed that we could improve the *SNR* by injecting *squeezed vacuum* into the unused port. Show that that in this case, the *SNR* is

$$SNR = \frac{\left\langle \hat{I}_{out} \right\rangle}{\sqrt{\left\langle \Delta\hat{I}^{2}_{out} \right\rangle}} \approx e^{r}\delta\phi\sqrt{\left\langle \hat{n}^{a}_{in} \right\rangle}\ ,$$

where *r* is the squeezing parameter. This is currently be implemented in the latest generation of LIGO.

http://www.nature.com/nphoton/journal/v7/n8/full/nphoton.2013.177.html