Physics 581, Quantum Optics II
Problem Set #2
Due: Tuesday February 18, 2014

Problem 1: Nonclassical light generation via the Kerr effect. (10 points)
In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\chi^{(3)}}{2} : \hat{I}^2 := \frac{\hbar\chi^{(3)}}{2} \hat{a}^{\dagger 2}\hat{a}^2 .$$

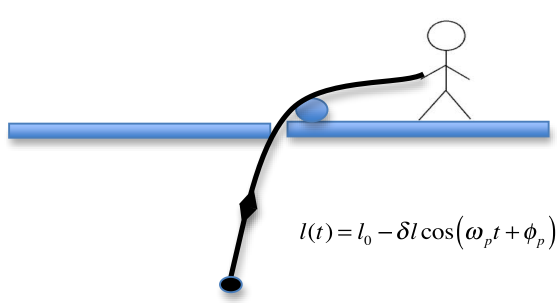
(a) Suppose we inject a strong coherent state into a nonlinear fiber with Kerr response. *Linearize* this Hamiltonian about the mean field via the substitution $\hat{a} = \alpha + \hat{b}$, and keep terms only up to quadratic order in $\hat{b}$ and $\hat{b}^\dagger$. Show that the resulting leads to squeezing.

(b) Now let's go beyond the linear approximation. Show that for a long time such that $\chi^{(3)} t = \pi$, the state becomes a Schrödinger cat, $\left(e^{i\pi/4}\left|-i\alpha\right\rangle + e^{-i\pi/4}\left|i\alpha\right\rangle\right)/\sqrt{2}$.

Note: Though in principle this is the solution, in practice this is not observed with light because losses and other noise sets in long before this kind of coherence can be established. More recently, the analogous experiment has been performed with Bose-Einstein condensates of atoms, which shows the same nonlinear dynamics (see Greiner *et al., Nature* **419**, 51-54 (5 September 2002)).

Problem 2: Parametric resonance (15 points)
Optical parametric amplification that we studied in the context of three-wave mixing in nonlinear optics is a general phenomenon in nonlinear dynamics know as *parametric resonance*. We study the basic problem here.

Consider an oscillator with a *time-dependent frequency* $\ddot{x} + \omega^2(t)x = 0$. For example, consider a pendulum whose length is periodically modulated



When $\delta l << l_0$ the pendulum oscillations satisfy the Mathieu equation

$$\ddot{x} + \left(\omega_0^2 + \varepsilon\cos(\omega_p t - \phi_p)\right)x = 0 \text{, where } \varepsilon = \frac{\delta l}{l_0}\omega_0^2 \text{ and } \omega_0 = \sqrt{g/l_0}$$

There is no general analytic solution to this problem. We can, however, solve this approximately. Our goal is to show that there a nonlinear resonance, at which point we exponential pump energy into the system.

(a) Write the general solution as $x = \mathrm{Re}\left(\alpha(t)e^{-i\omega_0 t}\right)$. For weak driving take $|\dot{\alpha}| << \omega_0|\alpha|$.

Show that

$$\dot{\alpha} - \dot{\alpha}^* e^{2i\omega_0 t} \approx -i\frac{\varepsilon}{4\omega_0}\left(\alpha e^{i\phi_P}e^{-i\omega_p t} + \alpha e^{-i\phi_P}e^{+i\omega_p t} + \alpha^* e^{-i\phi_P}e^{+i(\omega_P+2\omega_0)t} + \alpha^* e^{i\phi_P}e^{-i(\omega_P-2\omega_0)t}\right).$$

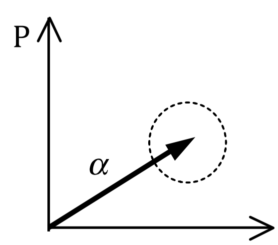
(b) This equation shows "parametric resonance" when $\omega_P = 2\omega_0$ (the term "parametric" comes from the idea that we were modulating a parameter in the original oscillator). Ignoring the rapidly oscillating term, show that at parametric resonance,

$$\dot{\alpha} \approx \zeta\alpha^*, \quad \zeta = re^{i2\theta} \text{ with solution } X_\theta(t) = e^{-r}X_\theta(0), \quad P_\theta(t) = e^{+r}P_\theta(0), \quad \alpha e^{-i\theta} \equiv X_\theta + iP_\theta$$

(find ζ, r, θ)

We are all familiar with this phenomenon. As a child on a swing, we pump our legs back and forth, effectively increasing or decreasing the length of the pendulum. If we pump at twice the natural frequency we amplify our motion, but only if we pump at the right phase! In nonlinear optics, the pump laser effectively changes the optical path length of the signal and can thus parametrically amplify the signal.

(c) Parametric resonance leads to *phase-sensitive amplification*. Consider a classical statistical distribution of initial complex amplitudes.



For the conditions such that $\theta = 0$, sketch the resulting output distribution. Comment.

Problem 3: Squeezing in the Heisenberg Picture (20 Points)

(a) Consider a "pure squeezing" Hamiltonian, $\hat{H} = \hbar\kappa^*\hat{a}^2 + \hbar\kappa\hat{a}^{\dagger 2}$. Show that the Heisenberg equations of motion and solutions are

$$\frac{d\hat{a}}{dt} = -i2\kappa\hat{a}^\dagger, \ \frac{d\hat{a}^\dagger}{dt} = +i2\kappa^*\hat{a} \Rightarrow \hat{a}(t) = \cosh(r)\hat{a}(0) - e^{i2\theta}\sinh(r)\hat{a}^\dagger(0). \text{ Find } r,\theta.$$

This is the Bogoliubov transformation, corresponding to parametric amplification.
Note: $\hat{a}(t) = \mu\hat{a}(0) - \nu\hat{a}^\dagger(0)$, with $|\mu|^2 - |\nu|^2 = 1$, and thus $r = \sinh^{-1}(|\nu|)$.

(b) Now consider a squeezing interaction in the present of a rotation (caused by a phase mismatch), $\hat{H} = \hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\kappa^*\hat{a}^2 + \hbar\kappa\hat{a}^{\dagger 2}$. Find the Heisenberg equations of motion and show that the solution is

$$\hat{a}(t) = \left[\cosh\Omega - i\frac{\Delta t}{\Omega}\sinh\Omega\right]\hat{a}(0) - \left[e^{i2\theta}\frac{r}{\Omega}\sinh\Omega\right]\hat{a}^\dagger(0)$$

where r,θ are as in part (a) and $\Omega = \sqrt{r^2 - (\Delta t)^2}$.

(c) Show that, as in part (a), in part (b), $\hat{a}(t) = \mu\hat{a}(0) - \nu\hat{a}^\dagger(0)$, with $|\mu|^2 - |\nu|^2 = 1$. This is generic result for any Hamiltonian that is quadratic in $\hat{a},\hat{a}^\dagger$ (more on this fact to come). Using this, show that this interaction leads to squeezing with squeezing parameter

$$\tilde{r} = \sinh^{-1}\left(\frac{r}{\Omega}\sinh\Omega\right) = \log\left(\frac{r}{\Omega}\sinh\Omega + \sqrt{1 + \left(\frac{r}{\Omega}\sinh\Omega\right)^2}\right)$$

(d) Show that only when we have perfect phase matching ($\Delta = 0$) do we achieve exponential growth with time (amplification) of one quadrature and deamplifaction (squeezing) of the other.

Problem 4: Some more boson Algebra (20 Points – extra credit)

(a) Show that the displacement operators are orthogonal according to the Hilbert-Schmidt inner product, $Tr\left(\hat{D}^\dagger(\alpha)\hat{D}(\beta)\right) = \pi\delta^{(2)}(\alpha-\beta)$.

Hint: Recall $Tr\left(\hat{A}\right) = Tr\left(\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|\hat{A}\right) = \int \frac{d^2\alpha}{\pi}\langle\alpha|\hat{A}|\alpha\rangle$

(b) We have shown that the Fourier transform of the displacement operators are

$$\hat{T}_\sigma(\alpha) \equiv \int \frac{d^2\beta}{\pi}\hat{D}_\sigma(\alpha)e^{\alpha\beta^*-\alpha^*\beta} = \pi\left\{\delta^{(2)}(\alpha-\hat{a})\delta^{(2)}(\alpha^*-\hat{a}^\dagger)\right\}_\sigma .$$

Show that $\hat{T}_{-1}(\alpha) = |\alpha\rangle\langle\alpha|$

(c) Show that for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, the Wigner function is

$$W(X,P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi}\psi^*\left(X+\frac{Y}{2}\right)\psi^*\left(X-\frac{Y}{2}\right)e^{-iPY} \quad \text{, where } W(X,P) = \frac{1}{2}W(\alpha) \ .$$

(d) Show that the Wigner function yields the correct marginals in X and P,

$$\int_{-\infty}^{\infty} dP\, W(X,P) = |\psi(X)|^2, \quad \int_{-\infty}^{\infty} dX\, W(X,P) = |\tilde{\psi}(P)|^2 ,$$

and for an arbitrary quadrature

$$\int_{-\infty}^{\infty} dP_\theta\, W(X,P) = |\tilde{\psi}(X_\theta)|^2$$