Physics 581, Quantum Optics II

Problem Set #3
Due: Thursday March 13, 2014

Problem 1: Properties of the Wigner function (10 Points)

(a) Show that the Wigner function we derived based on the operator ordered representation of the Weyl operator take standard form originally given by Wigner,

$$W(X,P)=\int \frac{dY}{2\pi} e^{-iPY} \left\langle X-Y/2 \middle| \hat{\rho} \middle| X+Y/2 \right\rangle = \int \frac{dY}{2\pi} e^{-iPY}\, \psi(X-Y/2)\psi^*(X+Y/2)\,,$$

where the last form applies only for a pure state.

(b) In standard statistics, given a joint probability distribution on many random variables, one defines the "marginal distributions" by integrating out the others.

$$\mathrm{P}(X)=\int dP\; W(X,P) \quad \mathrm{P}(P)=\int dX\; W(X,P)\,.$$

Show that these marginal are in fact the *correct* true marginals predicted by quantum mechanics (here QM gives true probability distributions).

(c) Suppose we have operators which are functions only of the quadratures $\hat{f}_1 = f_1(\hat{X}),\ \hat{f}_2 = f_2(\hat{P})$. Show that

$$\left\langle \hat{f}_1 \right\rangle = \int dX\, dP\, W(X,P)\, f_1(X) = \int dX\; \mathrm{P}(X)\, f_1(X),$$
$$\left\langle \hat{f}_2 \right\rangle = \int dX\, dP\, W(X,P)\, f_2(X) = \int dX\; \mathrm{P}(P)\, f_2(P),$$

and thus show, for example, the quantum uncertainties ΔX and ΔP are the respective rms widths of the Wigner function.

(d) Generalize (b) and (c) to the case of the rotated quadratures,

$$\hat{X}_\theta = \cos\theta\, \hat{X} + \sin\theta\, \hat{P},$$
$$\hat{P}_\theta = \cos\theta\, \hat{P} - \sin\theta\, \hat{X}.$$

Problem 2: Calculation of some quasiprobability functions (25 points)

(a) Find the *P. Q,* and *W* distributions for a thermal state

$$\hat{\rho} = \frac{e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_BT}}{Z}, Z = Tr(e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_BT}) = \text{partition function}$$

and show they are *Gaussian* functions. For example, you should find $P(\alpha) = \frac{1}{\pi\langle n\rangle}\exp\left(-\frac{|\alpha|^2}{\langle n\rangle}\right)$.

Show that these three distributions give the proper functions in the limit, $\langle n\rangle \to 0$, i.e. the vacuum.

(b) Find the *P. Q,* and *W* distributions squeezed state $|\psi\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$. In what sense is this state nonclassical?

(c) Find the Glauber-Sudharshan P-representation for a Fock state $|\psi\rangle = |n\rangle$. Comment.

(d) Consider a superposition state of two "macroscopically" distinguishable coherent states, $|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, $|\alpha_1 - \alpha_2| >> 1$, where $N = \left[2\left(1 + \exp\{-|\alpha_1 - \alpha_2|^2\}\right)\right]^{-1/2}$ is normalization.
This state is often referred to as a "Schrodinger cat", and is very nonclassical. Calculate the Wigner function, for the case $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$, with α real, and plot it for different values of $|\alpha_1 - \alpha_2| = 2\alpha$. Comment please.

(e) Calculate the marginals of the Schrödinger-cat Wigner function in *X* and *P* and show they are what you expect.

Problem 3: An Alternative Representation of the Wigner Function. (15 points)

We have shown that Wigner function could be expressed as

$$W(\alpha)=\frac{1}{\pi}Tr\left(\hat{\rho}\hat{T}(\alpha)\right)=\frac{1}{\pi}\left\langle\hat{T}(\alpha)\right\rangle, \text{ where } \hat{T}(\alpha)=\int\frac{d^2\beta}{\pi}\hat{D}(\beta)e^{\alpha\beta^*-\beta^*\alpha}$$

(a) Show that $\hat{T}(\alpha)=\hat{D}(\alpha)\hat{T}(0)\hat{D}^\dagger(\alpha)$.

(b) Show that $\hat{T}(0)=2(-1)^{\hat{a}^\dagger\hat{a}}$. (This is a tough problem. You may assume the answer and work backwards or try to find a direct proof).

Note: the operator $(-1)^{\hat{a}^\dagger\hat{a}}=\sum_n(-1)^n|n\rangle\langle n|=\int dX|-X\rangle\langle X|$ is the "parity operator" (+1 for even parity, -1 for odd parity). Thus we see that the Wigner function at the origin is given by the expected value of the parity.

$$W(0)=\frac{2}{\pi}Tr\left[\hat{\rho}(-1)^{\hat{a}^\dagger\hat{a}}\right]=\frac{2}{\pi}\sum_n(-1)^n\langle n|\hat{\rho}|n\rangle.$$

(c) Show that general expression

$$\hat{T}(\alpha)=2\hat{D}(\alpha)(-1)^{\hat{a}^\dagger\hat{a}}\hat{D}^\dagger(\alpha)=2\sum_n(-1)^n\hat{D}(\alpha)|n\rangle\langle n|\hat{D}^\dagger(\alpha),$$

$$\text{and thus } W(\alpha)=\frac{2}{\pi}\sum_n(-1)^n\langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle.$$

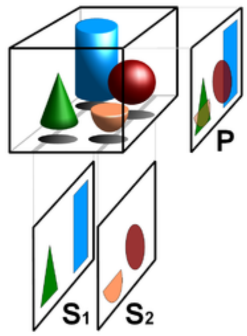
This expression provides a way to "measure" the Wigner function. One displaces the state to the point of interest, $\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)$, one then measures the photon statistics $p_{n\alpha}=\langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle$. Putting this in the parity sum gives $W(\alpha)$ at that point!

This is a form a quantum-state reconstruction, also know as "quantum tomography," which we will study in the next problem.

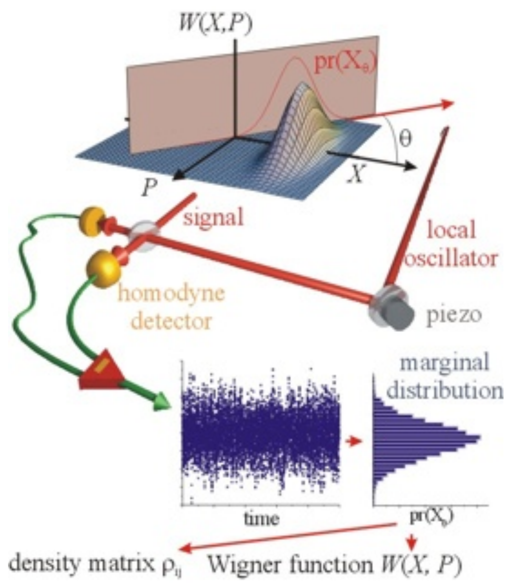
The measurement of displaced number states in quantum optics is not easy. However, it is much more straightforward in the context of measurements the vibrational state of trapped ions, as was performed in the group of Dave Wineland in one of the first demonstrations of a negative Wigner function when the ion was prepared in an n=1 Fock state (see Leibfreid *et al*., PRL **77**, 4281 (1996)).

Problem 4: Quantum Tomography (10 points)

Form Wikipedia: "*Tomography* refers to imaging by sections or sectioning." The idea is to reconstruct a three dimensional object by a series of projections onto different planes.



The term "quantum tomography," was coined by the group of Raymer (Smithey *et al.*, PRL 70, 1244 (1993)) in the context of reconstructing the Wigner function of a light mode. This was based on the work of Vogel and Risken (PRA **40**, 2847 (1989)) who showed that the Wigner function was related to the measurements of the marginals – the projections of the Wigner function onto a plane.



We learned that a homodyne detector measures a quadrature of the field with the plane determined by the phase of the local oscillator. Thus, the measurement outcome sampled from the probability distribution, $pr(X_\theta) = \langle X_\theta | \hat{\rho} | X_\theta \rangle$, where $|X_\theta\rangle$ are the eigenstates of $\hat{X}_\theta = (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/\sqrt{2} = \hat{X}\cos\theta + \hat{P}\sin\theta$

The goal of quantum tomography is to invert and determine $\hat{\rho}$ (or equivalently $W(\alpha)$) given $\{pr(X_\theta), \forall\theta\}$.

(a) Prove the following Lemma.

Given $\delta(x)\delta(y) = \int \frac{dk_x dk_y}{(2\pi)^2} e^{-ik_x x} e^{-ik_y y}$, show that by transforming the polar coordinates in the Fourier plane,

$$\delta(X - X')\delta(P - P') = \frac{1}{(2\pi)^2}\int_0^\infty k\,dk \int_0^{2\pi} d\theta\, e^{-ik(X_\theta - X'_\theta)} = \frac{1}{(2\pi)^2}\int_{-\infty}^\infty |k|\,dk \int_0^\pi d\theta\, e^{-ik(X_\theta - X'_\theta)}.$$

(b) Starting with the trivial identity,

$$W(X,P) = \int dX'\,dP'\delta(X - X')\delta(P - P')W(X',P'),$$

and the result of part (a), show that the Wigner function can be obtained form the marginals by,

$$W(X,P) = \int_0^\pi d\theta \int_{-\infty}^\infty dX'_\theta\, pr(X'_\theta)K(X_\theta - X'_\theta),$$

where the integral kernel

$$K(X) = \frac{1}{4\pi^2}\int_{-\infty}^\infty dk|k|e^{-ikX} \ .$$

This is known as the "*Radon transformation*," originally written in the context of classical tomographic image processing.

We see immediately that integral kernel is not well defined and blows up. This means that the Radon transformation is numerically unstable. In addition, it is not robust to the physical case that we have only a discrete set of measurements of $\{pr(X_\theta)\}$, and the detection is not perfect.

In the intervening decades since the original experiments in quantum tomography, the reconstruction process has been refined with more sophisticated estimation schemes based on statistical interference. This has spurred a new line of research in quantum information science regarding the question of what is required to reconstruct a quantum state given finite measurement resources and noise.

(c) Extra credit (5 points). Show that

$$W(X,P) = -\frac{1}{2}\int_0^\pi d\theta\, \mathcal{P}\int_{-\infty}^\infty dX'_\theta \left(\frac{1}{X_\theta - X'_\theta}\right)\frac{d}{dX'_\theta} pr(X'_\theta).$$

where $\mathcal{P}$ is the Cauchy Principal Value. This expression shows how finite knowledge of $\{pr(X_\theta)\}$ limits the Radon transform, since we do not know the derivative of $pr(X_\theta)$, and need to interpolate if we only have finite points.

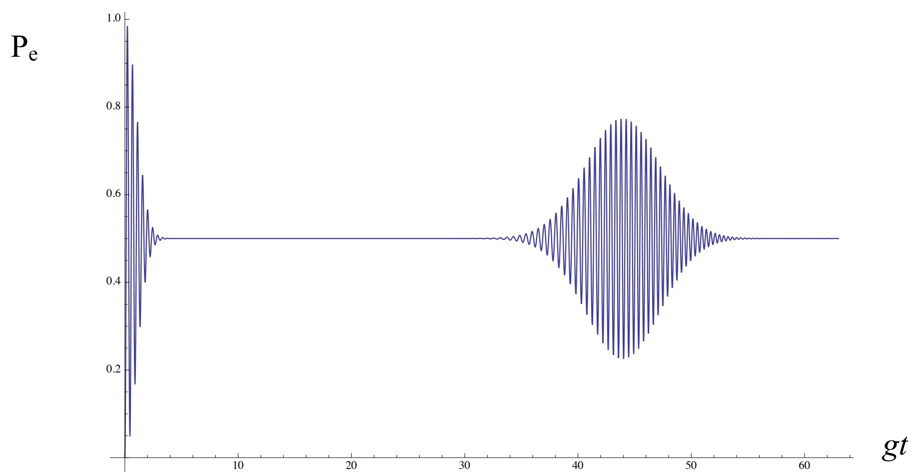
Problem 5: Entanglement and the Jaynes-Cummings Model (30 points)

One the most fundamental paradigms in quantum optics is the coupling of a two-level atom to a single mode of the quantized electromagnetic field. In the rotating wave approximation, this is governed by the Jaynes-Cummings model (JCM),

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_0 \frac{\hat{\sigma}_z}{2} + \hbar g\left(\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-\right).$$

This is a bipartite system with tensor product Hilbert space for the atom and field, $\mathcal{H}_{AF} = h_A \otimes h_F$, where h_A is the two-dimensional Hilbert space of the two-level atom, and h_F is the infinite dimensional Hilbert space of the harmonic oscillator that describes the mode. The goal of this problem is to understand the entanglement between the atom and mode, generated by the JCM.

Last semester, we studied how this leads to collapse and revival of Rabi oscillation that follows from an initial product state with the field in a coherent state and the atom in, e.g., the ground state $|\Psi(0)\rangle_{AF} = |g\rangle_A \otimes |\alpha\rangle_F$. The probability to find the atom in the excited state oscillates as shown (here for $\langle n \rangle = |\alpha|^2 = 49$)



The collapse is due to the variation of the quantum Rabi oscillations with different number; the revival is uniquely a quantum effect arising from the discreteness of the quantized field, occurring at a time $gt_r \approx 2\pi\sqrt{\langle n \rangle}$ for large $\langle n \rangle$.

(a) Show that the state at time *t* the joint state takes the form

$$|\Psi(t)\rangle_{AF} = |C(t)\rangle_F \otimes |g\rangle_A + |S(t)\rangle_F \otimes |e\rangle_A$$

where $|C(t)\rangle_F = \sum_{n=0}^{\infty} c_n \cos(\sqrt{n}gt)|n\rangle$, $|S(t)\rangle_F = -i\sum_{n=0}^{\infty} c_{n+1} \sin(\sqrt{n+1}gt)|n\rangle$, $c_n = (\alpha^n / \sqrt{n!})e^{-|\alpha|^2/2}$.

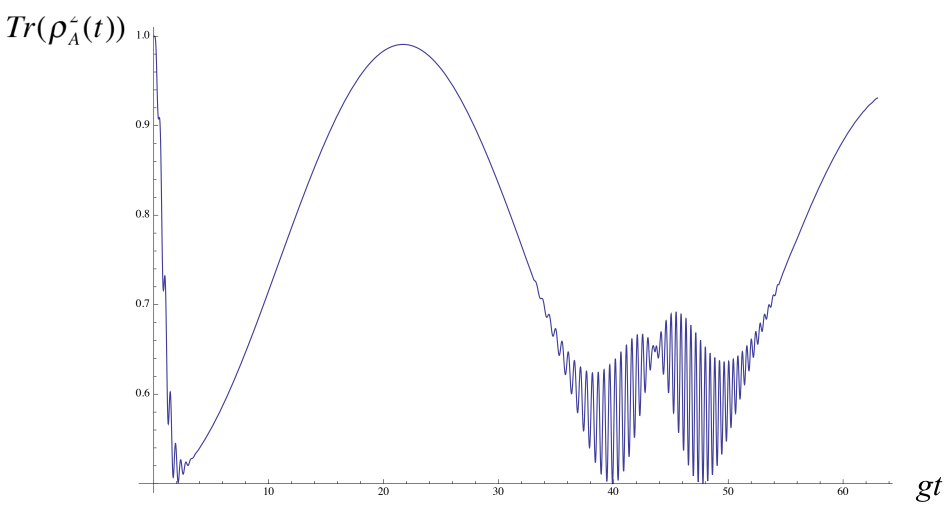
Note $|C(t)\rangle_F, |S(t)\rangle_F$ are not normalized, nor are they orthogonal.

(b) Show that the marginal state of the atom in the $\{|g\rangle, |e\rangle\}$ basis is

$$\hat{\rho}_A(t) = \begin{bmatrix} \langle C(t)|C(t)\rangle & \langle C(t)|S(t)\rangle \\ \langle S(t)|C(t)\rangle & \langle S(t)|S(t)\rangle \end{bmatrix} = \frac{1}{2}\left(\hat{1} + \vec{Q}(t)\cdot\hat{\vec{\sigma}}\right).$$

Write an expression for Bloch vector $\vec{Q}(t)$.

(c) Write the purity of the marginal (a measure of the entanglement between the atom and field), in terms of the Bloch vector. Numerically calculate this and plot as a function of time for $\langle n\rangle = |\alpha|^2 = 49$. Your graph should look like



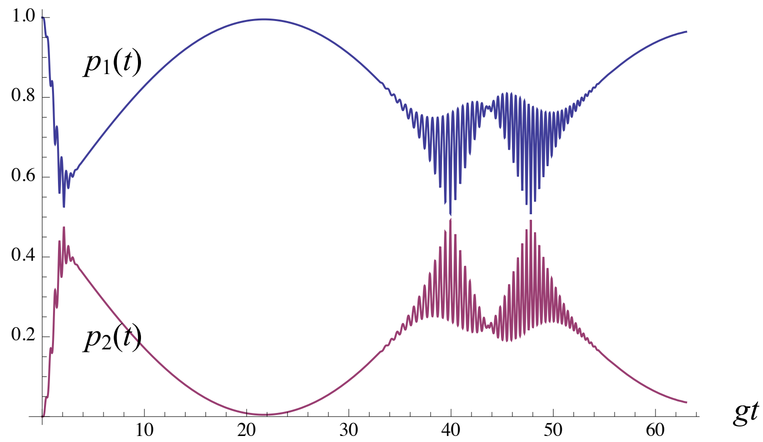
This plots shows a few surprising features. During the collapse the atom and field become highly entangled, as indicated by the rapid degree in the atomic purity. However, at half the revival time, $gt_r/2 \approx \pi\sqrt{\langle n\rangle}$, when the inversion looks to be flat, the purity returns to near unity, indicating that the atom and field become *separable*. The atom and field then become re-entangled. When the Rabi oscillations once again revive, the purity again increases, but nowhere near to unity. Our goal now is to use the Schmidt decomposition to understand this.

(d) Given the initial pure state of the joint system and the unitary evolution according to the JCM, we know that at all times we can express the state in terms of Schmidt decomposition.

$$|\Psi(t)\rangle_{AF} = \sum_{\mu=1}^{2}\sqrt{p_u(t)}\,|u_\mu(t)\rangle_A \otimes |v_\mu(t)\rangle_A .$$

Note, even though the field mode is infinite dimensional, the maximum Schmidt number is 2.

Express the two values of $p_\mu(t)$ in terms of the Bloch vector $\vec{Q}(t)$. Calculate numerically at plot as function of time. Your graphs should look like the following:



Comment on this and what it means for the entanglement.

(e) We can find the Schmidt vectors by the following procedure.

- Find the atomic Schmidt vectors $\left\{\left|u_\mu(t)\right\rangle_A\right\}$ as the eigenvectors of the marginal state $\hat{\rho}_A(t)$ in the standard basis $\{|g\rangle,|e\rangle\}$.
- Using $|\Psi(t)\rangle_{AF} = |C(t)\rangle_F \otimes |g\rangle_A + |S(t)\rangle_F \otimes |e\rangle_A = \sum_{\mu=1}^{2} \sqrt{p_u(t)}\left|u_\mu(t)\right\rangle_A \otimes \left|v_\mu(t)\right\rangle_A$, find an expression for the two Schmidt vectors of the field $\left\{\left|v_\mu(t)\right\rangle_F\right\}$ in terms of $|C(t)\rangle_F, |S(t)\rangle_F, p_u(t)$.

(f) We can see the (approximate) separation between atom and field at half the revival time for large $\langle n \rangle$ as follows. Show that in this limit,

$$g\sqrt{n+1}t_r/2 \approx g\sqrt{n}t_r/2 + \pi/2,\ \ c_{n+1} \approx e^{-i\phi}c_n, \text{ where } c_n = (\alpha^n/\sqrt{n!})e^{-|\alpha|^2/2} \text{ and } \alpha = \sqrt{\langle n \rangle}e^{i\phi}\ .$$

Using this, show that

$$|\Psi(t_r/2)\rangle_{AF} \approx \left(|g\rangle_A - ie^{-i\phi}|e\rangle_A\right) \otimes |C(t_r/2)\rangle_F\ .$$

Thus we see that the system is separable, with the atom in an equal superposition depending on the phase of the coherence state.

(g) Extra credit (5 points): More generally show that if $|\Psi(0)\rangle_{AF} = \left(a|g\rangle_A + b|e\rangle_A\right) \otimes |\alpha\rangle_F$

$$|\Psi(t_r/2)\rangle_{AF} \approx \left(|g\rangle_A - ie^{-i\phi}|e\rangle_A\right) \otimes \left(a|C(t_r/2)\rangle_F + b|S(t_r/2)\rangle_F\right)$$

This result shows that *regardless of the atomic initial state*, at half the revival time, the atom goes to the same state. The information about the initial atomic superposition is transferred to the field in a kind of "swap gate." For large α, the two field states are macroscopically distinguishable. This is kind of "Schrödinger cat".

Problem 6: Gaussian States in Quantum Optics (35 points)

The set of states whose quadrature fluctuations are Gaussian distributed about a mean value is an important class in quantum optics. These states have Gaussian Wigner functions. In this problem, we explore Gaussian states, their relationship to squeezing, and the canonical algebra of phase space.

Consider a field of *n-modes*, with quadrature defined by an ordered vector:

$$\mathbf{Z} = (X_1, P_1, X_2, P_2, \ldots, X_n, P_n).$$

The operators associated with these quadratures satisfy a set of canonical commutators relations that can be written compactly as,

$$\left[\hat{Z}_i, \hat{Z}_j\right] = \frac{i}{2}\Sigma_{ij}, \text{ where } \Sigma = \bigoplus_{k=1}^{n}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix.}$$

We define an "inner product" in phase space as $\left(\mathbf{Z}\middle|\mathbf{Q}\right) = Z_i \Sigma_{ij} Q_j$ (summed over repeated indices through this problem).

(a) Show that the phase space displacement operator can be written

$$\hat{D}(\mathbf{Z}) = \exp\left\{-i\left(\mathbf{Z}\middle|\hat{\mathbf{Z}}\right)\right\}$$

A *Gaussian state* is one whose Wigner function is a Gaussian function on phase space. Recall the characteristic function of a quantum state is defined $\chi(\mathbf{Z}) = Tr\left(\hat{\rho}\hat{D}(\mathbf{Z})\right)$.

The general form of the characteristic function for a Gaussian state with is:

$$\chi(\mathbf{Z}) = \exp\left\{-\frac{1}{2}\left(\mathbf{Z}\middle|\mathbf{C}\middle|\mathbf{Z}\right) + i\left(\mathbf{d}\middle|\mathbf{Z}\right)\right\}.$$

Where C_{ij} is known as the covariance matrix, and d_i is a real vector.

(b) Show that: $\left\langle \hat{Z}_i \right\rangle = d_i$, and $\frac{1}{2}\left\langle \Delta\hat{Z}_i\Delta\hat{Z}_j + \Delta\hat{Z}_j\Delta\hat{Z}_i \right\rangle = C_{ij}$, where $\Delta\hat{Z}_i \equiv \hat{Z}_i - \left\langle \hat{Z}_i \right\rangle$.

Hint: Recall how moments are found from the characteristic function.

The Gaussian state is thus determined by the mean position in phase space and the covariance of all the fluctuations.

(c) Find the Wigner function for a state with the general form of the characteristic function.

Let us restrict our attention to Gaussian states with zero mean (the mean is irrelevant to the statistics and can always be removed via a displacement operation). Consider now unitary transformations on the state. A particular class of transformations is the set that act as linear canonical transformations, i.e.

$$\hat{U}^\dagger \hat{Z}_i \hat{U} = S_{ij}\hat{Z}_j \text{, where } S_{ij} \text{ is a symplectic matrix, defined by } S^T \Sigma S = \Sigma .$$

A unitary map on the state transforms the state according to

$$\chi(\mathbf{Z}) \Rightarrow \chi'(\mathbf{Z}) = Tr\left(\hat{U}\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\right) = Tr\left(\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\hat{U}\right).$$

(d) Show that for a symplectic transformation, the characteristic function transforms as

$$\chi(\mathbf{Z}) \Rightarrow \chi(\mathbf{SZ})$$

and thus the action of the unitary is to *preserve the Gaussian statistics, by* transforming covariance matrix as $\mathbf{C} \Rightarrow \mathbf{S}^T\mathbf{CS}$.

(e) Show that the following operations preserve Gaussian statistics:

- Linear optics: $\hat{U} = \exp\left(-i\theta_{ij}\hat{a}_i^\dagger \hat{a}_j\right)$
- Squeezing: $\hat{U} = \exp\left(\zeta_{ij}^*\hat{a}_i\hat{a}_j - \zeta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger\right)$

(f) For each of these, show how the covariance matrix of the Gaussian transforms.

(g) Starting with the vacuum (a Gaussian state) we apply the squeezing operator above. Show that the symplectic transformation on the covariant matrix leads to the expected result.