Physics 581: Quantum Optics II
Lecture 1: Review

In Physics 566, Quantum Optics I, we introduced some key concepts, which we review here.

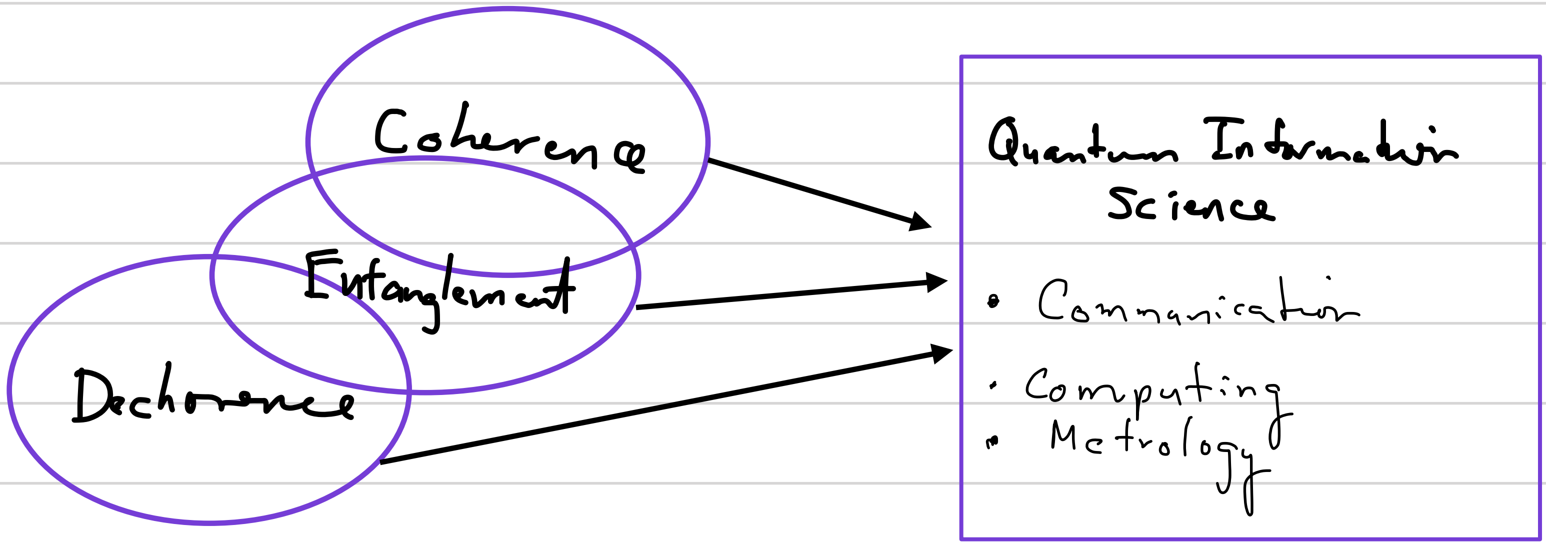
Overview: Coherence, Entanglement, Decoherence, and Quantum Information

The central subject of quantum optics is coherence in quantum system as induced by or intrinsic in optical (or more generally) electromagnetic fields. Coherence is the capacity of a system to exhibit interference. Interference is a "wave phenomenon" arising due to the principle superposition: the superposition of wave solutions to a linear differential equation is also a solution. In classical wave theory, this gives rise to phenomena such as Young's double slit and Michaelson interferometers. In quantum physics, "indistinguish processes" interfere. Each process is typically associated with "quanta" or "particles." Thus, quantum coherent phenomena are intimately connected with wave/particle duality. The quantum coherent phenomena of particles and waves will be a central theme this semester.

Such phenomena take on a new dimension when they are associated with multiple quanta. In that case, the interference phenomena associated with the history of two or more particles has no classical wave interference analog. We have see one such example of great importance in quantum optics: the Hanbury-Brown & Twiss phenomenon. The generalization to so-called "nonclassical light" will be another central topic of this semester. Of particular importance in understanding multi-quanta interference is the idea of entanglement, its relation to the foundations of quantum mechanics itself. Is quantum mechanics compatible with a "local realistist, objective description of nature? Quantum optics has been the physical platform in which these seemingly metaphysical questions have been put to the nuts-and-bolts test of laboratory experiment.

While quantum coherence can lead to unique and powerful phenomena, particularly in the realm of "many-body physics" with entanglement, such coherence rarely survives into the macroscopic realm of our daily existence. The process by which a system goes from coherent – having the capacity to exhibit interference – to incoherent – without interference between said processes – is known as decoherence, and is an essential ingredient in understanding the quantum world, much as the thermodynamic equilibrium is essential to understanding the macroscopic classical world. Indeed, whereas thermodynamics arises in "open systems," where a small set of degrees of freedom are coupled to a reservoir, decoherence arises in "open quantum systems." Beyond its practical relevance, the study of decoherence and open quantum systems dynamics is an essential ingredient in "measurement theory." This too is an issue in the foundations of quantum mechanics – perhaps THE issue of the foundation. How are measurement outcomes, which are generically random, extracted from the microscopic deterministic predictions of the Schrödinger equation? In this semester we will study both the nuts-and-bolts on open quantum system dynamics and its relationship to understanding quantum measurement.

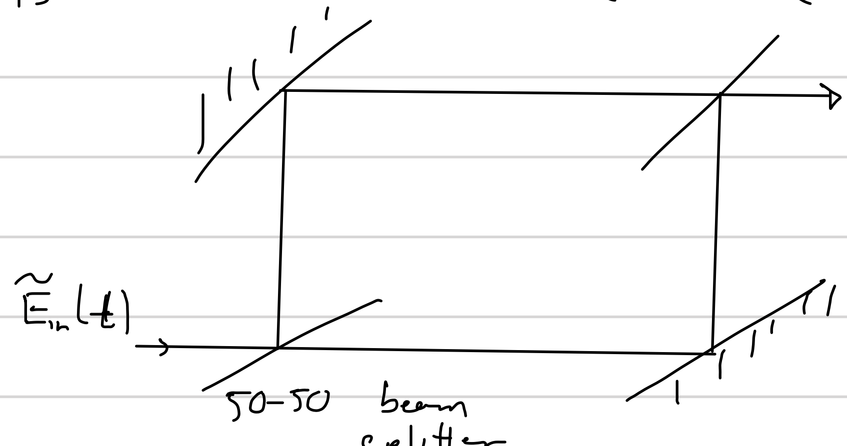
Historically, whereas the founders of quantum theory emphasized the ways in which quantum physics was "restricted" relative to classical physics, i.e., the uncertainty principle limiting the precision with which we could simultaneously predict the outcome of a measurement of position and momentum, in the 1990s, physicists, mathematicians, and computer scientists came to the understanding that exactly the opposite was true. Quantum physics, far from being a paler version of classical physics, gives nature more power than ever conceived. In particular, quantum physics opens the door to devices that can process information in ways that no classical information processor can. With its focus on the study of the foundations of quantum mechanics in the 1980's with the development of experiments to control measure individual quanta, and a theoretical framework to analyze and think out such systems, the subdiscipline of Quantum Optics became the natural forum in which to study the implementation of quantum information processing devices. Quantum opticians have populated the new discipline that has emerged, quantum information science (QIS), and these physicists have been some of its founders. We will study the applications of quantum optics to QIS as time permits.



Coherence:

Standard paradigm: Two path interferometer (e.g. Mach-Zender)

Classical: Wave interference



$$\tilde{E}_{out}(t) = \frac{\tilde{E}_{in}(t - \frac{l_1}{c}) + \tilde{E}_{in}(t - \frac{l_2}{c})}{2}$$

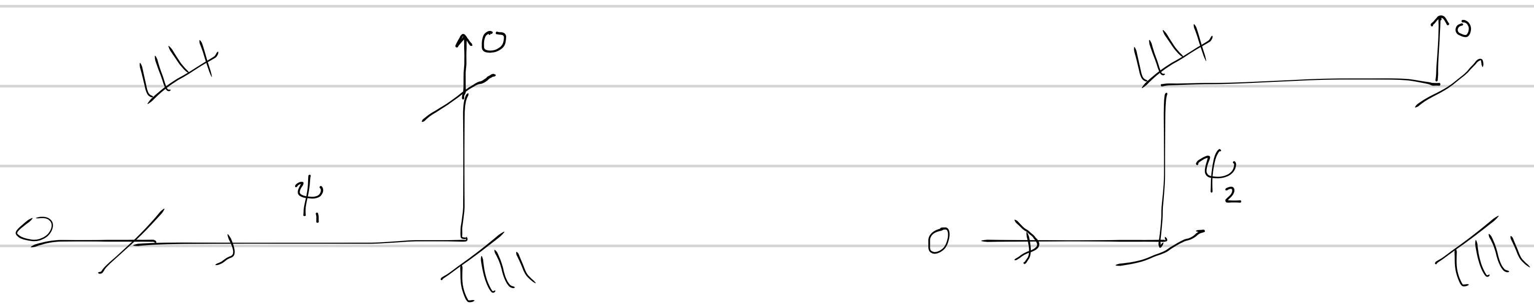
$\leftarrow$ ensemble average over probability distribution = time ave (ergodic)

$$I_{out} = \overline{\tilde{E}^*_{out}(t)\,\tilde{E}_{out}(t)}$$

$$= \frac{1}{2}\Big(I_{in} + \mathrm{Re}\big(\underbrace{\overline{\tilde{E}^*(t-\tau)\,\tilde{E}(t)}}_{\text{Field-field (auto) correlation } G^{(1)}(\tau)}\big)\Big) \qquad \tau = \frac{l_2 - l_1}{c}$$

$$G^{(1)}(\tau) = \int d\{\alpha_k\}\,P(\{\alpha_k\})\;\tilde{E}^*(t-\tau)\,\tilde{E}(t), \qquad \tilde{E}(t) = \sum_k E_{0k}\,\alpha_k\,e^{-i\omega_k t}$$

Quantum: Alternative paths



In first order coherence, each photon "interfere only with itself"
First order interference involves "single particle physics" $\Rightarrow$ linear optical phenomenona.

Density matrix and quantum coherence

$\hat{\rho} \equiv$ State of the system (Density operator)

Pure state: $\hat{\rho} = |\psi\rangle\langle\psi|$ (note ray in Hilbert space $e^{i\theta}|\psi\rangle \Rightarrow$ same $\hat{\rho}$)

Statistical mixture: $\hat{\rho} = \sum_i P_i |\psi_i\rangle\langle\psi_i|$, $P_i \geq 0$ (classical probability)

Ensemble decomposition (not unique)

Important Properties:

- Hermitian $\hat{\rho}^\dagger = \hat{\rho}$, Postive $\hat{\rho} \geq 0$ (All eigenvalues ≥ 0)
- Normalization $\mathrm{Tr}(\hat{\rho}) = \mathrm{Tr}(\sum_i P_i |\psi_i\rangle\langle\psi_i|) = \sum_i P_i = 1$
- Diagonal matrix element: $\langle a|\hat{\rho}|a\rangle = \mathrm{Tr}(\hat{\rho}\, |a\rangle\langle a|) =$ Probability to find outcome "a" in a measurement, e.g., $\{|a\rangle\}$ are eigenvectors of Hermitian "observable" $\hat{A}$
- Off-diagonal matrix element: $\langle a|\hat{\rho}|a'\rangle$ = Coherence: The ability of the system to exhibit interference between the two outcomes, a and a'.
- Purity: $\frac{1}{d} \leq \mathrm{Tr}(\rho^2) \leq 1$ d: dimension of Hilbert space

maximall mixed $\hat{\rho} = \frac{1}{d}\mathbb{1}$ Pure state $\hat{\rho} = |\psi\rangle\langle\psi|$

E.g. $d=\infty$ "Continuous Variable:" Position $\{|x\rangle\}$ (e.g. non-relativistic particle)

- $\langle x|\hat{\rho}|x\rangle = \sum_i P_i \langle x|\psi_i\rangle\langle\psi_i|x\rangle = \sum_i P_i \psi_i^*(x)\psi_i(x) = \sum_i P_i |\psi_i(x)|^2$

  = Probability density to find particle around x

- $\langle x'|\hat{\rho}|x\rangle = \sum_i P_i \psi_i^*(x')\psi_i(x') = \overline{\psi^*(x')\psi(x)}$

  (Quantum analog of coherence function $\overline{E(x',t')E(x,t)}$)

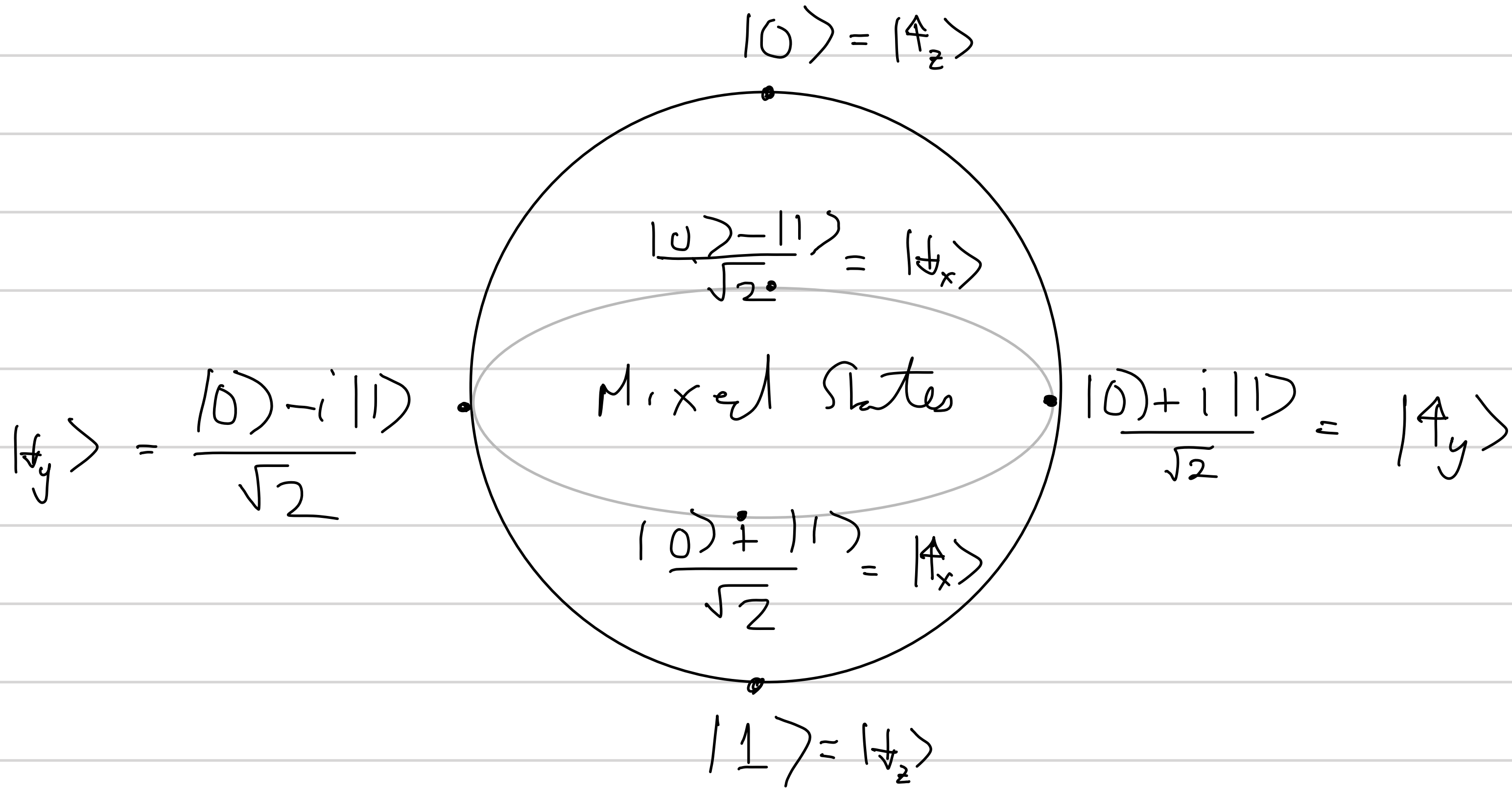
E.g. $d=2$ "Qubit" Standard basis $\{|0\rangle, |1\rangle\}$

Isomorphic to spin-$\frac{1}{2}$: $|0\rangle = |\uparrow_z\rangle$, $|\downarrow_z\rangle = |1\rangle$

Pauli operators $\hat{\sigma}_x = \underbrace{|0\rangle\langle 1|}_{\hat{\sigma}_+} + \underbrace{|1\rangle\langle 0|}_{\hat{\sigma}_-}$, $\hat{\sigma}_y = \frac{|0\rangle\langle 1| - |1\rangle\langle 0|}{i}$, $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$\hat{\rho} = \frac{\hat{\mathbb{1}} + \vec{Q}\cdot\hat{\vec{\sigma}}}{2} \qquad \vec{Q} = \mathrm{Tr}(\rho\,\hat{\vec{\sigma}}) \quad : \text{Bloch vector}$$

<u>Bloch Sphere (Ball)</u> : Space of all quantum states of qubit



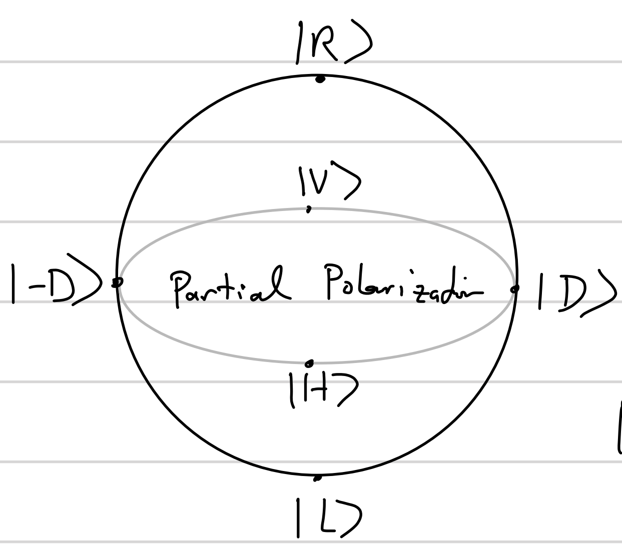
Example enconding of a qubit : Polarization state of a photon

$|0\rangle \equiv |R\rangle$: Right hand Circular $\qquad$ $|1\rangle = |L\rangle$: Left Hand Circular

R $\circlearrowleft \rightarrow \vec{k}$ $\qquad$ L $\circlearrowright \rightarrow \vec{k}$

$|H\rangle$ = Horitonal linear $= \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, $\qquad$ $|V\rangle$ = Vertical linear $= \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$|D\rangle = 45^\circ$ linear $= \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ $\qquad$ $|\bar{D}\rangle = -45^\circ$ linear $= \frac{|0\rangle - i|1\rangle}{\sqrt{}}$



Polarization Poincaré Sphere

Quantized Electromagnetic Field

A core part (some would say *the* part) of quantum optics in the quantum nature of the electromagnetic field itself. A key question is to identify those phenomenon that are "essentially quantum," which we call "nonclassical light" and which can be described by classical statistical optics with perhaps a quantum description of matter. To properly treat the quantized electromagnetic field requires the formalism of quantum field theory. Here we consider low energy (optical and lower freq.) photons.

Quantum Field theory — Wave degrees of freedom : Normal modes of oscillation

Quantum Field theory — particle degrees of freedom : Quanta of excitations

Quantum Field theory $\Updownarrow$ many-body physics

Classical field decomposition in terms of normal modes

$$\vec{E}(\vec{r},t) = \sum_{\vec{k},\mu}{}' \mathcal{E}_{\vec{k}}\, \alpha_{\vec{k},\mu}(t)\, \vec{u}_{\vec{k},\mu}(\vec{r}) + \text{c.c.}$$

$\mathcal{E}_{\vec{k}}$: normalization; $\vec{u}_{\vec{k},\mu}(\vec{r})$: Normal modes depending on boundary conditions

E.g. Plane waves in 3D with periodic boundary conditions

$$\vec{u}_{\vec{k},\mu}(\vec{r}) = \frac{1}{\sqrt{V}}\vec{\epsilon}_{\vec{k},\mu}\, e^{i\vec{k}\cdot\vec{x}} \qquad \vec{k} = \left(\frac{2\pi}{L_x}m_x,\ \frac{2\pi}{L_y}m_y,\ \frac{2\pi}{L_z}m_z\right)$$

Transverse Polarization $\vec{k}\cdot\vec{\epsilon}_{\vec{k},\mu} = 0$

μ = labels two orthogonal polarization in plane $\perp$ to $\vec{k}$

In a cavity: Standing wave Modes

$$\vec{u}_{\vec{k},\mu}(\vec{r}) \sim \frac{1}{\sqrt{V}}\vec{\epsilon}_{\vec{k},\mu}\cos(kz)\, e^{-\frac{|\vec{r}_\perp|^2}{2w^2}} \qquad k = \left(m\frac{\pi}{L},\ m = 1, 2, \ldots\right)$$

Here longitudinal modes

$$\alpha_{\vec{k}\mu}(t) = \alpha_{\vec{k}\mu}(0)\, e^{-i\omega_k t} = \text{normal mode complex amplitude}$$

Quantize: $\alpha_{k,\mu} \Rightarrow \hat{a}_{\vec{k},\mu}$ quantum annihilation operator.

Free field, periodic boundary conditions (Heisenberg picture)

$$\hat{\vec{E}}(\vec{r},t) = \underbrace{\sum_{\vec{k},\mu}' \sqrt{\frac{2\pi\hbar\omega_k}{V}}\, \vec{\epsilon}_{\vec{k},\mu}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}\, \hat{a}_{\vec{k},\mu}}_{\hat{\vec{E}}^{(+)}(\vec{r},t)} + \underbrace{h.c.}_{\hat{\vec{E}}^{(-)}(\vec{r},t)}$$

$\hat{a}^\dagger_{\vec{k},\mu}$ ($\hat{a}_{\vec{k},\mu}$) creates (annihilates) a photon - quantum of excitation in the plane wave mode $\vec{\epsilon}_{\vec{k}\mu} e^{i(\vec{k}\cdot\vec{r}-\omega_k t)}$

$$[\hat{a}_{\vec{k},\mu}, \hat{a}^\dagger_{\vec{k}',\mu'}] = \delta_{\vec{k},\vec{k}'}\delta_{\mu\mu'} \quad \text{(different modes commute)}$$

Hilbert Space: Fock space $\mathcal{H}_{Fock}$: spanned by $\{\underbrace{|n_1, n_2, \dots n_k, \dots\rangle}_{\text{"Fock states"}} \mid n_k = 0, 1, 2, \dots \infty\}$

k ← composite index $(\vec{k},\mu)$

$$\mathcal{H}_{Fock} = \bigotimes_{n=1}^{\infty} h_{n_k} \qquad h_{n_k} = L_2(\mathbb{R}_1)$$: Hilbert space of square normalizable functions

h_{n_k} spanned by $\{|n_k\rangle \mid n_k = 0, 1, 2 .. \infty\}$ $\quad |n_k\rangle = \frac{(\hat{a}^\dagger_k)^{n_k}}{\sqrt{n_k!}}|0\rangle$ ← vacuum

= "Simple Harmonic oscillator" Hilbert space

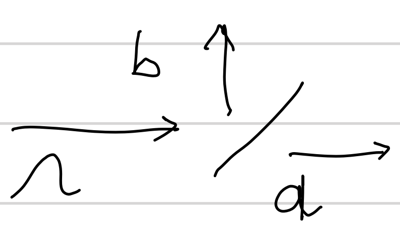
$|n_k\rangle$ = n_k photons in the mode k; eg. plane wave mode

Single photon state not necessarily in one mode:

$$|1_\psi\rangle = \hat{a}^\dagger[\psi]\,|0\rangle = \sum_{\vec{k}} \psi(\vec{k})\,\hat{a}^\dagger_{\vec{k}}|0\rangle$$

$\psi(\vec{k})$ = single photon momentum space wavefunct.

E.g. single photon at a beam splitter



single photon input

output
$$|\psi\rangle_{out} = (t\,\hat{a}^\dagger_{k_a} + r\,\hat{a}^\dagger_{k_b})|0\rangle$$

transmission, reflection amplitudes

The single photon is in a superposition of two plane waves $e^{i\vec{k}_a\cdot\vec{r}}$ and $e^{i\vec{k}_b\cdot\vec{r}}$

Note: In second quantized notation $|\Psi_{out}\rangle = r|1_a, 0_b\rangle + t|0_a 1_b\rangle$

Is this an entangled state?

__Two-photon state__ One photon in momentum space wave packet $\psi(\vec{k})$ and another in wave packet $\phi(\vec{k})$

$$|\Psi_2\rangle = \frac{\hat{a}^\dagger[\psi]\hat{a}^\dagger[\phi]\,|0\rangle}{\sqrt{2(1+|\langle\phi|\psi\rangle|^2)}}$$

This is the second quantized form of the state

In "first quantization" $\Psi_2(\vec{k}_1,\vec{k}_2)$ is the joint probability amplitude to detect one photon with momentum $\vec{k}_1$ and one with momentum $\vec{k}_2$

$$\Psi_2(\vec{k}_1,\vec{k}_2) = \langle 1_{\vec{k}_1}, 1_{\vec{k}_2}|\Psi_2\rangle = \langle 0|\hat{a}_{\vec{k}_1}\hat{a}_{\vec{k}_2}\hat{a}^\dagger[\psi]\hat{a}^\dagger[\phi]|0\rangle \frac{1}{\sqrt{1+\langle\psi|\phi\rangle}}$$

$$= \langle 0|[\hat{a}_{\vec{k}_1}\hat{a}_{\vec{k}_2}\hat{a}^\dagger[\psi]\hat{a}^\dagger[\phi]]|0\rangle$$

$$= \frac{1}{\sqrt{2(1+|\langle\psi|\phi\rangle|^2)}}\left(\psi(\vec{k}_1)\phi(\vec{k}_2) + \phi(\vec{k}_1)\psi(\vec{k}_2)\right)$$

Having used $[a_{\vec{k}}, \hat{a}^\dagger[\psi]] = \psi(\vec{k})$ etc.

This is the expected result for a __two boson__ wave function, which is symmetric under exchange of identical particles. The second quantization formalism takes this into account automatically.

__Wave degrees of freedom__ (continuous variable (CV) quantum optics)

For a given mode (drop label) $\hat{a}$ is the quantized "complex amplitude".

$$\alpha = \frac{X+iP}{\sqrt{2}} \leftarrow \text{normalization convention}$$

$$X = \sqrt{2}\,\mathrm{Re}(\alpha) = \frac{\alpha+\alpha^*}{\sqrt{2}}, \quad P = \sqrt{2}\,\mathrm{Im}(P) = \frac{\alpha-\alpha^*}{i\sqrt{2}} \quad \text{(Quadratures)}$$

Classical SHO: $\alpha(t) = \alpha(0)e^{-i\omega t}$ (complex amplitude)

$$X(t) = \sqrt{2}\,\mathrm{Re}(\alpha(t)) = X(0)\cos\omega t + P(0)\sin\omega t$$

Quantum: $\hat{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}}$, $\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}$, $\hat{P} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}$ (Quadrature operators)

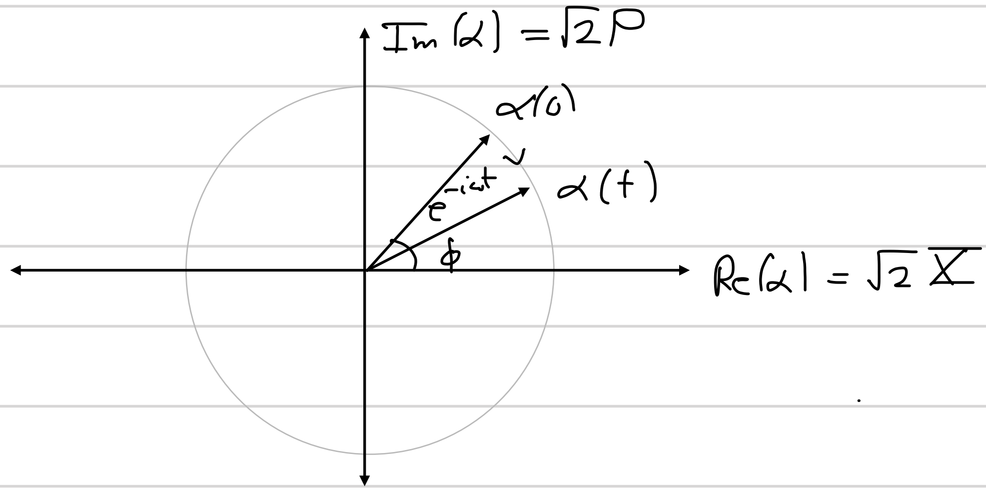
$[\hat{X}, \hat{P}] = i$ (canonical commutation relation)

Uncertainty principle: $\Delta X \Delta P \geq \frac{1}{2}$

Minimum uncertainty, e.g. vacuum $\Delta X_0^2 = \Delta P_0^2 = \frac{1}{2}$

Amplitude and Phase: $\alpha = |\alpha| e^{i\phi} = \sqrt{\alpha^* \alpha}\, e^{i\phi}$ (Polar decomposition)

$\alpha(t) = |\alpha|\, e^{-i(\omega t - \phi)} \Rightarrow X(t) = |\alpha| \cos(\phi - \omega t) = |\alpha| \cos(\omega t - \phi)$



Quantum: $\hat{n} = \hat{a}^\dagger \hat{a}$ = # operator $= \sum_{n=0}^{\infty} n\, |n\rangle\langle n|$

"Amplitude" $\sqrt{\hat{n}} = \sum_{n=0}^{\infty} \sqrt{n}\, |n\rangle\langle n|$

Number eigenstates = Energy eigenstates of the simple Harmonic oscillator: Free field Hamiltonian $\hat{H} = \frac{1}{8\pi} \int d^3x \, (\hat{E}^2 + \hat{B}^2) = \sum_{k,\mu} \hbar\omega_k \left(\hat{a}^\dagger_{k\mu} \hat{a}_{k\mu} + \frac{1}{2}\right)$ Zero point (often reset to zero).

Single mode: $\hat{H} = \hbar\omega\, \hat{a}^\dagger \hat{a}$, Time evolution $\hat{U} = e^{-i\omega t \hat{n}}$

Heisenberg evolution $\hat{U}^\dagger(t)\, \hat{a}\, \hat{U}(t) = \hat{a}\, e^{-i\omega t}$: rotation in phase space

Number operator: Generator of "phase displacements" (rotation)

Energy eigenstate $\Rightarrow$ Stationary state $\Rightarrow$ rotationary symmetric in phase space

$\Rightarrow$ Definite photon number $\Leftrightarrow$ completely uncertain phase.

Number and phase are "conjugate" variable. A quantum state with a well defined phase necessarily has an uncertainty in number.

To understand the "quantum phase" of the oscilator we consider the polar decomposition of $\hat{a} = \sum_{n=0}^{\infty} \sqrt{n}\, |n\rangle\langle n+1|$

$$\hat{a} = \widehat{e^{i\phi}} \sqrt{\hat{n}}\ , \qquad \hat{a}^\dagger = \sqrt{\hat{n}}\ \widehat{e^{i\phi}}^\dagger$$

where $\widehat{e^{i\phi}} = \hat{a} \sqrt{\hat{n}}^{-1} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$, $\widehat{e^{i\phi}}^\dagger = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$

Note $\widehat{e^{i\phi}}^\dagger \widehat{e^{i\phi}} = \sum_{n=0}^{\infty} |n+1\rangle\langle n+1| = \hat{\mathbb{1}} - |0\rangle\langle 0|$

$\Rightarrow \widehat{e^{i\phi}}$ is not unitary $\Rightarrow$ there does not exist a Hermitian phase operator $\hat{\Phi}$. Nonetheless, there exist "phase" states that are eigenstates of $\widehat{e^{i\phi}}^\dagger$: Define $|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle$ (equal superposition of all $|n\rangle$)

$$\widehat{e^{i\phi}}\, |e^{i\phi}\rangle = \sum_{n=1}^{\infty} e^{in\phi} |n-1\rangle = e^{i\phi} \sum_{n=0}^{\infty} e^{in\phi} |n\rangle = e^{i\phi} |e^{i\phi}\rangle$$

these states form a resolution of the identity $\int \frac{d\phi}{2\pi} |e^{i\phi}\rangle\langle e^{i\phi}| = \hat{\mathbb{1}}$

We see that phase is the "generator of number displacements." Thus, we see that number and phase are conjugate. In some sense. The case of number a phase is more subtle because there is no Hermitian phase operator. The uncertainty relation between n and ϕ does not follow from the commutator of $\hat{n}$ and $\hat{\phi}$. Mathematically the problem is that $\hat{n}$ is bounded from below at $n=0$. Nonetheless, for $n \gg 1$, we have approx.

$$[\hat{n}, \hat{\Phi}] \approx i \qquad \Delta n \Delta \phi \geq \frac{1}{2}$$

More formally, an ideal phase measurement is a so-called POVM (which we will study later), corresponding to the resolution into (over complete) phase states $\int d\phi\, \hat{E}_\phi = \mathbb{1}$ $\quad \hat{E}_\phi = \frac{1}{2\pi} |e^{i\phi}\rangle\langle e^{i\phi}|$. We can generalize the uncertainty principle to include POVMs, not just orthogonal bases.

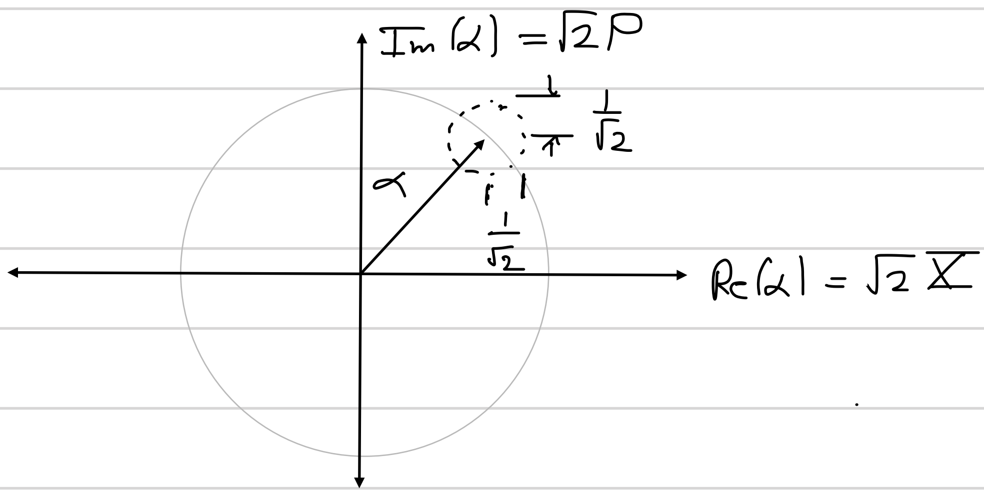
Coherent States (Quasi classical states)

A state of the harmonic oscillator that plays a central role in quantum optics are the quasiclassical states, which Glauber called "coherent states". The defining property is $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$: Eigenstate of $\hat{a}$, and thus the quantum mechanical state with that is closest to having a well-defined complex amplitude, consisent with the uncertainty principle

Properties:

$\langle\alpha|\hat{X}|\alpha\rangle = \sqrt{2}\, Re(\alpha)$, $\quad \langle\alpha|\hat{P}|\alpha\rangle = \sqrt{2}\, Im(\alpha)$,

$\langle\alpha|\Delta X^2|\alpha\rangle = \langle\alpha|\Delta P^2|\alpha\rangle = \frac{1}{2}$: minimum uncertainty



Number fluctuations:

$$\langle\hat{n}\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$$

$$\langle\Delta n^2\rangle = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 = |\alpha|^2 = \langle\hat{n}\rangle$$

Generally: $\quad |\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad c_n = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}$

$$\Rightarrow \quad P_n = |c_n|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2} = \frac{\langle n\rangle^n}{n!} e^{-n}$$: Poisson distribution

Phase fluctuations $\quad P(\phi) = \frac{1}{2\pi} |\langle e^{i\phi}|\alpha\rangle|^2 = \frac{1}{2\pi} \left|\sum_\alpha \frac{|\alpha|^n}{\sqrt{n!}} e^{in(\phi-\phi_0)}\right|$, where $\alpha = |\alpha| e^{i\phi_0}$

$$\langle\phi\rangle = \phi_0 \qquad \Delta\phi \sim \frac{1}{|\alpha|} = \frac{1}{\sqrt{n}} = \frac{1}{\Delta n}, \qquad \Delta\phi\,\Delta n \sim 1$$

Time evolution: $\quad \hat{U}(t)|\alpha\rangle = e^{-i\omega t \hat{a}^\dagger\hat{a}}|\alpha\rangle = |\alpha(t)\rangle = |\alpha e^{-i\omega t}\rangle$

Coherent state remain coherent state for all time following classical trajectory.

The Poisson statistics represent the minimum fluctuation in photon number associated with a perfectly stable classical wave of definite intensity. Mixed states, associated with classical statistical fluctuations in the intensity add additional number fluctuations.

"Natural light" (thermal state): Bose-Einstein Statistics

$$\hat{\rho} = \sum_n P_n |n\rangle\langle n| \,, \qquad P_n = \frac{1}{\langle n\rangle + 1}\left[\frac{\langle n\rangle}{\langle n\rangle+1}\right]^n$$

$$\left(\text{In thermal equilibrium } \langle n\rangle = \frac{1}{e^{\hbar\omega/k_B T}-1}\right)$$

Continuous variable representation: Statistical mixture of coherent states

$$\hat{\rho} = \int d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle\alpha|$$

Glauber-Sudarshan P-representation

See homework
$$\begin{cases} P(\alpha) = \dfrac{1}{\pi \bar{n}} e^{-\frac{|\alpha|^2}{\bar{n}}} : \text{Gaussian amplitude fluctuations} \\ \langle\alpha\rangle = 0, \quad \left[\Delta|\alpha|^2\right]^2 = \langle n\rangle^2 \end{cases}$$

$$\Delta n^2 = \langle \hat{n}^2\rangle - \langle\hat{n}\rangle^2 = \langle \hat{a}^\dagger\hat{a}\,\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2$$
$$= \langle \hat{a}^\dagger\hat{a}^\dagger\,\hat{a}\,\hat{a}\rangle + \underbrace{\langle\hat{a}^\dagger\hat{a}\rangle}_{\text{term arising from the comutator (quantum fluct.)}} - \langle\hat{a}^\dagger\hat{a}\rangle^2$$

$$= \mathrm{Tr}\left[\hat{\rho}\,(\hat{a}^{\dagger 2}\hat{a}^2)\right] - \left(\mathrm{Tr}(\hat{\rho}\,\hat{a}^\dagger\hat{a})\right)^2 + \langle\hat{n}\rangle$$

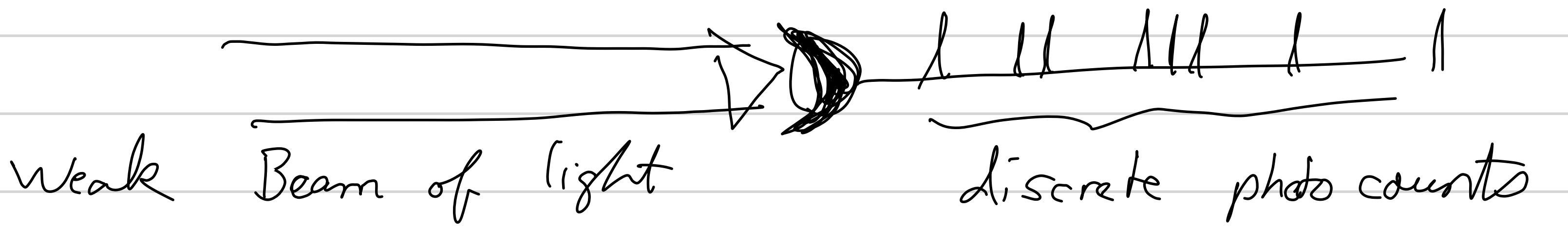
$$= \int d\alpha\, P(\alpha)\,|\alpha|^4 - \left(\int d\alpha\, P(\alpha)\,|\alpha|^2\right)^2 + \langle\hat{n}\rangle$$

$$= \underbrace{\left[\Delta|\alpha|^2\right]^2}_{\text{Wave noise}} + \underbrace{\langle\hat{n}\rangle}_{\text{Particle noise}}$$

If $\Delta n^2 < \langle\hat{n}\rangle$ (Sub Poissonian)
Nonclassical

Semiclassical theory of photon counting

A photon counter detects "clicks" through the photoelectric effect that causes the liberation of photo-electrons from a material.



These discrete events and all properties of the photoelectric effect are perfectly well predicted in a semiclassical theory in which the electromagnetic field is treated classically, not the matter is treated quantumly. The key assumption is that the matter's energy levels are quantized, and thus can only absorb in quantized steps and secondly, that these absorption processes are random and uncorrelated. This leads to the conclusion that, for a perfectly well defined intensity there will be fluctuations in the number of photocounts, which we call <u>shot noise</u>. The fluctuations are Poissonian, described by "Mandel's formula"

$$P(n,T) = \frac{\overline{n(I(t),T)}^{\,n}}{n!} e^{-\overline{n(I(t),T)}}, \text{ where } \overline{n(I(t),T)} = \eta \frac{A}{\hbar\omega_0} \int_0^T I(t)\,dt$$

where $I(t)$ is the beam intensity @ω_0 and η is a constant depending on "quantum efficiency" for absorbing a photon. When the intensity has classical statistical fluctuations

$$P(n,T) = \int dI \; P(I)\, P(n,T)$$

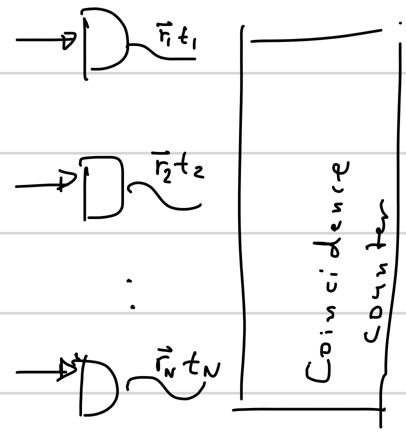
In the semiclassical theory, the minimum fluctuations are Poissonian, and excess noise on the classical intensities results in excess fluctuation in photo counts $\Rightarrow$ super Poissonian. Quantum mechanically there are fluctuations in the <u>arrival of photons</u>, but this is perfectly well describe by a semiclassical model.

Photon-Photon Correlations and higher-order interference

Glauber theory of photon coincidence:

Given N-photon counters, the probability of N-fold coincidence

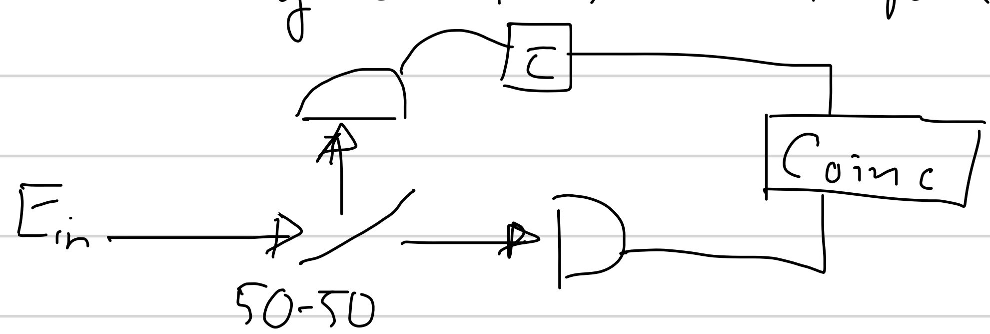
$\hat{I}(\vec{r}_1 t_1) = \hat{E}^{(-)}(\vec{r}_1 t_1)\,\hat{E}^{(+)}(\vec{r}_1\, t_1)$



$$P_{\text{coincidence}} \propto \langle : \hat{I}(\vec{r}_1, t_2)\, \hat{I}(\vec{r}_2, t_2) \cdots \hat{I}(\vec{r}_N, t_N) : \rangle \quad \leftarrow \text{Normal ordering}$$

$$= \langle \hat{E}^{(-)}(\vec{r}_1, t_1)\, \hat{E}^{(-)}(\vec{r}_2, t_2) \cdots \hat{E}^{(-)}(\vec{r}_N t_N)\, \hat{E}^{(+)}(\vec{r}_1 t_1)\, \hat{E}^{(+)}(\vec{r}_2 t_2) \cdots \hat{E}^{(+)}(\vec{r}_N t_N) \rangle$$

Example: Hanbury-Brown Twiss Temporal coherence

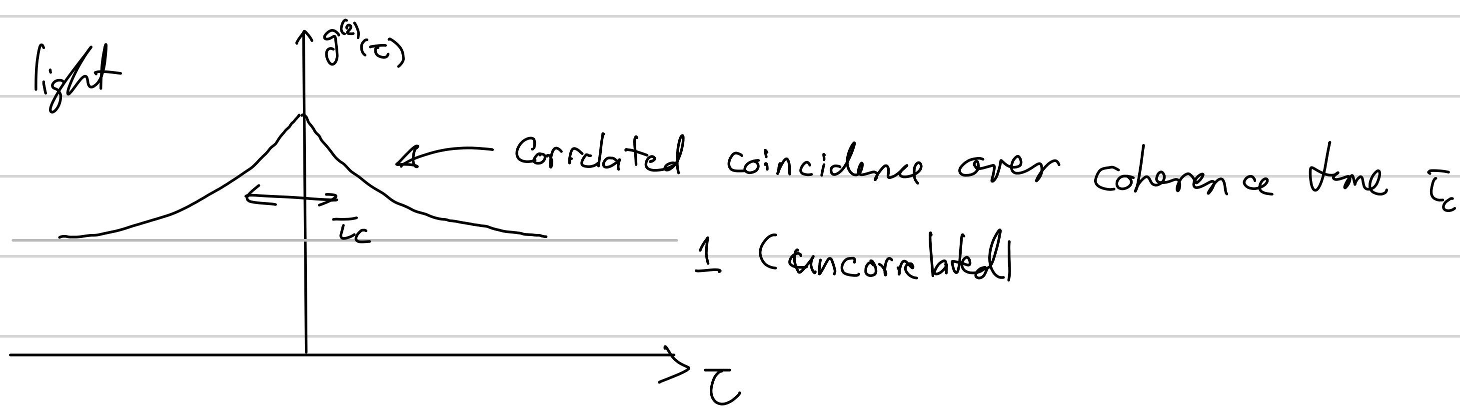


$\langle : \hat{I}(\tau)\, \hat{I}(0) : \rangle$
$= \langle \hat{E}^{(-)}(\tau)\, \hat{E}^{(-)}(0)\, \hat{E}^{(+)}(\tau)\, \hat{E}^{(+)}(0) \rangle$

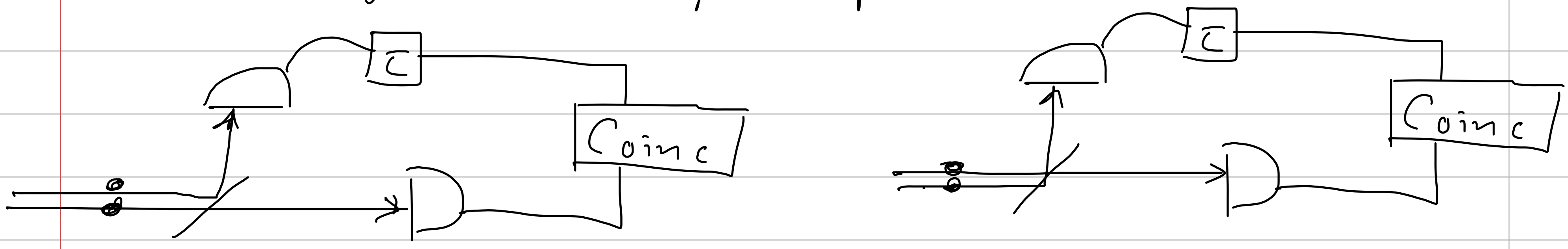
Photon counters, symmetrically placed on opposite output ports of a 50-50 beam splitter

$$P_{\text{coincidence}} \propto \langle : \hat{I}(\tau)\, \hat{I}(0) : \rangle, \qquad P_{\substack{\text{statistical}\\ \text{independent}}} = \langle : \hat{I} : \rangle^2$$

Thermal light



Indistinguishable two-photon process interfere



Photon-bunching and antibunching:

Define: $g^{(2)}(\tau) \equiv \dfrac{\langle : \hat{I}(\tau)\, \hat{I}(0) : \rangle}{\langle \hat{I} \rangle^2}$: $\dfrac{\text{Tendency of photons to arrive separated by } \tau}{\text{Statistical independent arrival}}$

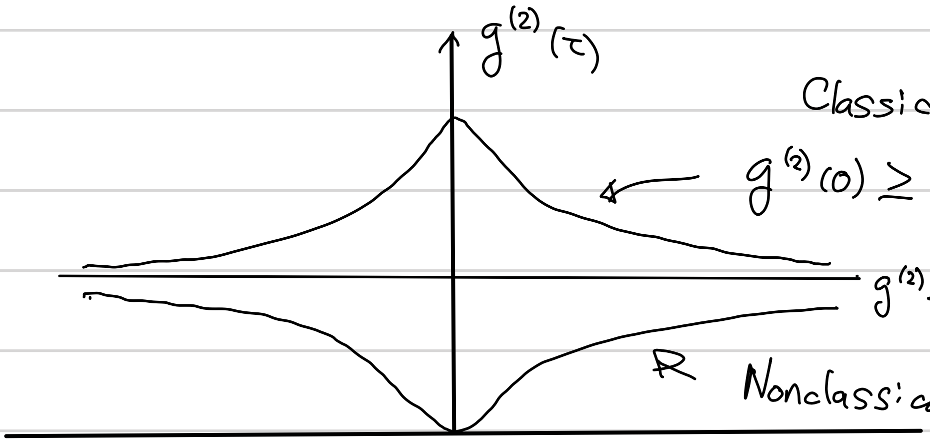
Photon bunching $g^{(2)}(0) > g^{(2)}(\tau)$ (Photons arrive in "clumps")
Photon antibunching $g^{(2)}(0) < g^{(2)}(\tau)$ (Photon arrival spread out)

It follows from Cauchy-Schwartz:

If $\hat{\rho} = \int d^2\{\alpha_k\}\, P(\{\alpha_k\})\, |\{\alpha_k\}\rangle\langle\{\alpha_k\}|$, $P(\{\alpha_k\}) \geq 0$

Then $g^{(2)}(0) \geq g^{(2)}(\tau)$:
(Statistically independent if coherent, bunched if noisy intensity)

If $g^{(2)}(0) < g^{(2)}(\tau)$: Photon-antibunching
$\Rightarrow$ Nonclassical Light

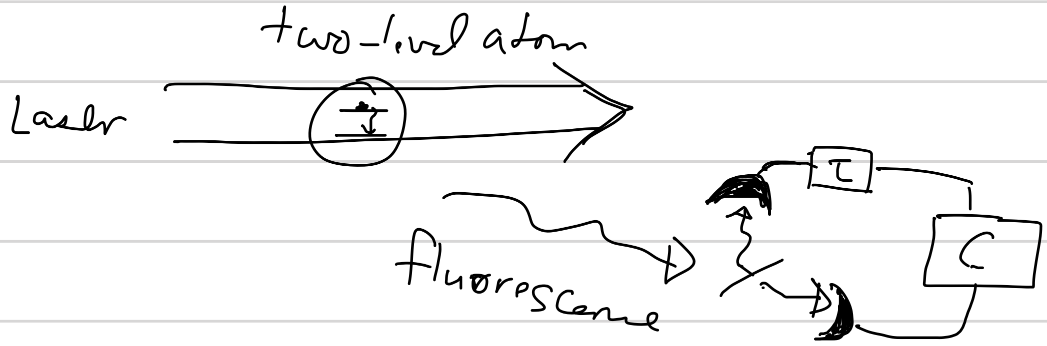


Classical light: (bunched)
$g^{(2)}(0) \geq g^{(2)}(\tau)$

$g^{(2)} = 1$ (coherent state)

Nonclassical light (antibunched)
$g^{(2)}(0) \leq g^{(2)}(\tau)$

Example: Resonance fluorescence from a single atomic emitter



If one photon counter clicks, we immediately measured that the atom jumped to the ground state. It take time for the atom to be reëxcited – the atom never emits two photons of this color at one time $\Rightarrow$ Photons are never arriving at the same time $\Rightarrow$ antibunching!