Lecture 5: Introduction to Entanglement

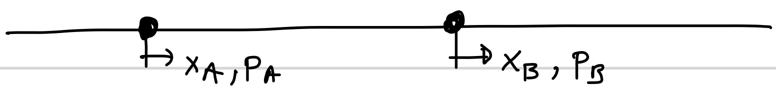
Our focus in the study of classical vs. nonclassical light has been on a single mode of the electromagnetic field. The nonclassicality was associated with the impossibility of formulating a the prediction of measurement outcome using a classical probability distribution. One of the most stark examples of a nonclassical phenomenon is entanglement associated with the correlation between different degrees of freedom. Entanglement has long fascinated students of quantum theory. It addresses the very heart of the question of whether quantum randomness describable by "local hidden variables," as first brought to the fore in famous Einstein-Padolfsky-Rosen (EPR) paradox, and the quantitative test defined by Bell's inequality.

The first test's of Bell's inequalities were done with photons. Quantum optics then became the natural forum in which to perform experimental explorations of the foundations of quantum mechanics. Both the theoretical framework for understanding these tests, and the experimental techniques needed for controlling and measuring individual quantum systems, were spurred by the desire for deeper understanding of the foundations. This was at the heart of the 2012 Nobel Prize in Physics to David Wineland & Serge Haroche. Moreover, this work laid the foundations for the application to quantum information science. The nonclassical phenomenon at the foundation of quantum mechanics, beyond being a deeply interesting metaphysical problem, also has application to new information processing paradigms: computers, algorithms, cryptography, communication.

Much of quantum optics has "morphed" into quantum information science because of this shared history and the potential application of quantum optical systems in the next generation of quantum information processors.

Quantum Mechanics of Multiple Degrees of Freedom

Consider the simplest classical system with two degrees of freedom - two point particles moving on a line:



The quantum mechanical wave function for the joint probability amplitude

$$\Psi(x_A, x_B) \Rightarrow P(x_A, x_B) = |\Psi(x_A, x_B)|^2$$

We define the marginal probability distributions

$$P(x_A) = \int_{-\infty}^{\infty} dx_B \, P(x_A, x_B) \, , \quad P(x_B) = \int_{-\infty}^{\infty} dx_A \, P(x_A, x_B)$$

Question: Does $P(x_A) = |\psi_A(x_A)|^2$, $P(x_B) = |\psi_B(x_A)|^2$ for some ψ_A and ψ_B?

Answer: Iff $\Psi(x_A, x_B) = \psi_A(x_A)\psi_B(x_B) \Rightarrow P(x_A, x_B) = P(x_A)P(x_B)$

$\Rightarrow$ x_A + x_B are uncorrelated $\rightarrow$ Ψ is a "product state": separable

If $\Psi(x_A, x_B) \neq \psi_A(x_A)\,\psi_B(x_B)$ for some ψ_A + ψ_B the states is said to be entangled

Note we assumed the joint state of A + B was pure. If the state of joint state of A+B is mixed, the definition and quantification of entanglement becomes much more complicated. We will not treat that problem here.

Tensor product

The formal structure of Hilbert space for multiple degrees of freedom is the tensor product. Consider two degrees of freedom (two parts = bipartite).

$$\mathcal{H}_{AB} = h_A \otimes h_B$$

$\otimes$ ← tensor (kronecker) product

$\mathcal{H}_{AB}$ ↑ Joint Hilbert space of the joint system

h_A, h_B ↑ Hilbert spaces of the two "subsystems"

if $|\psi\rangle_A \in h_A$ $|\phi\rangle_B \in h_B$ Define $|\Psi_{AB}\rangle \equiv |\psi\rangle_A \otimes |\psi\rangle_B \in \mathcal{H}_{AB}$

product state

Basis $\{|e_i\rangle_A \mid i = 1, 2, \cdots, d_A\}$ for h_B $\{|f_j\rangle_B \mid j = 1, 2, \cdots, d_B\}$

(d_A, d_B ↑ dimension of h_A, h_B repectively)

Basis for $\mathcal{H}_{AB}$: $|i,j\rangle \equiv |e_i\rangle_A \otimes |f_j\rangle_B$: $\mathrm{Dim}(\mathcal{H}_{AB}) = d_A d_B$

Representation: $\alpha_i \equiv \langle e_i|\psi\rangle_A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{d_A} \end{bmatrix}$ $\quad \beta_j \equiv \langle j|\phi\rangle_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{d_B} \end{bmatrix}$

$$\Rightarrow \gamma_{ij}^{AB} \equiv \langle i,j|\Psi\rangle_{AB} = \left({}_A\langle e_i| \otimes {}_B\langle f_j|\right)\left(|\psi\rangle_A \otimes |\phi\rangle_B\right) = \langle e_i|\psi\rangle_A \langle f_j|\phi\rangle_B$$

$$\Rightarrow \gamma_{ij}^{AB} = \alpha_i \beta_j$$: tensor product, outer product, Kronecker product

$$\gamma_{ij}^{AB} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{d_A} \end{bmatrix}_A \otimes \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{d_B} \end{bmatrix}_B = \begin{bmatrix} \alpha_1 \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{d_B} \end{bmatrix} \\ \alpha_2 \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{d_B} \end{bmatrix} \\ \vdots \\ \alpha_{d_A} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{d_B} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \vdots \\ \alpha_1\beta_{d_B} \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \vdots \\ \alpha_2\beta_{d_B} \\ \vdots \\ \alpha_{d_A}\beta_1 \\ \alpha_{d_A}\beta_2 \\ \vdots \\ \alpha_{d_A}\beta_{d_B} \end{bmatrix}_{AB}$$

Tensor product of operators

If $\hat{M}^{(A)}$ is an operator on h_A and $\hat{N}^{(B)}$ is and operater on h_B.

Define $\hat{O}^{(AB)} = \hat{M}^{(A)} \otimes \hat{N}^{(B)}$ on $\mathcal{H}_{AB}$ s.t. $\hat{O}^{(AB)} |\psi_A\rangle \otimes |\phi_B\rangle = \hat{M}^{(A)}|\psi\rangle_A \otimes \hat{N}^{(B)}|\phi\rangle_B$

Matrix representation: $O^{AB}_{(ij)(i'j')} = \langle ij | \hat{O}^{(AB)} | i'j' \rangle = N^{(A)}_{ii'} \, M^{(B)}_{jj'}$

Example: Two qubits $d_A = d_B = 2$

$\hat{O}^{AB} = \hat{\sigma}^A_x \otimes \hat{\sigma}^B_z$ Standard basis: $\underbrace{|i,j\rangle}_{\text{binary for } 1,2,3,4}$ $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$

$$\hat{O}^{AB} \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Marginal State ("reduced density operator)

Joint probability distribution: $p^{AB}_{ij} = |\langle ij | \Psi \rangle_{AB}|^2 = \langle ij | \hat{\rho}^{AB} | ij \rangle$

$\hat{\rho}_{AB} = |\Psi\rangle_{AB}\langle\Psi|$ (pure state)

Marginal $p^A_i = \sum_j p^{AB}_{ij} = \sum_j {}_A\langle i| \otimes {}_B\langle j| \hat{\rho}^{AB} |j\rangle_B \otimes |i\rangle_A = {}_A\langle i| \hat{\rho}^A |i\rangle_A$

where $\hat{\rho}^A = \underbrace{\mathrm{Tr}_B(\hat{\rho}^{AB}) = {}_B\langle j| \hat{\rho}^{AB} |j\rangle_B}_{\text{partial trace}}$ = marginal density operator (often the "reduced" density op)

Partial trace: $\hat{\rho}^{AB} = \sum_{ij\, i'j'} \rho_{(ij)(i'j')} |ij\rangle\langle i'j'| = \sum_{ij\, i'j'} \rho_{(ij)(i'j')} |i\rangle_A\langle i'| \otimes |j\rangle_B\langle j'|$

$\Rightarrow \hat{\rho}^A = \mathrm{Tr}_B(\hat{\rho}^{AB}) = \sum_j \underbrace{\rho_{(ij)(ij)}}_{= \mathrm{Tr}_B(\rho_{(ij)(i'j')})} |j\rangle_B\langle j'|$

Entangled States

- A pure state in $\mathcal{H}_{AB}$ is separable if a product state: $|\Psi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$ for some $|\psi\rangle_A \in \mathcal{H}_A$, $|\phi\rangle_B \in \mathcal{H}_B$. Otherwise $|\Psi\rangle_{AB}$ is entangled.

Example: Two qubits. Consider the state

$$|\Psi\rangle_{AB} = \frac{1}{2}\left(|\uparrow\uparrow\rangle + i|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + i|\downarrow\downarrow\rangle\right)$$

Though at first glance this looks entangled, in fact it is not.

$$|\Psi\rangle_{AB} = |\uparrow_x\rangle_A \otimes |\uparrow_y\rangle_B : \quad |\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right), \quad |\uparrow_y\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$$

While it might be possible to see this by eye for two qubits, for higher dimensional systems this becomes increasingly more difficult. We thus seek a more systematic method for determining the entanglement and, moreover, to quantify the degree of entanglement.

Classical vs. Quantum Correlations

How is entanglement different from classical correlations? Consider the singlet state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}|\uparrow_z\rangle_A \otimes |\downarrow_z\rangle_B - \frac{1}{\sqrt{2}}|\downarrow_z\rangle_A \otimes |\uparrow_z\rangle_B$$

The measure outcomes on spin-A and spin-B are correlated. Suppose Alice measures has access to spin-A. Her marginal state $\hat{\rho}_A = \frac{1}{2}\hat{\mathbb{1}}$, the completely mixed state. Thus no matter what she measures, she gets a random result: $\frac{1}{2}$ probability of outcome. Suppose she measures $\hat{\sigma}_z^A$, and finds $|\uparrow_z\rangle_A$ (with probability $\frac{1}{2}$). If she knows the joint state, then she can assign a state to Bob (who has access to spin-B), after her measurement

$$|\psi\rangle_B\Big|_{|\uparrow_z\rangle_A} = \frac{{}_A\langle\uparrow_z|\Psi_{AB}\rangle}{\|{}_A\langle\uparrow_z|\Psi_{AB}\rangle\|} = |\downarrow_z\rangle_B \Rightarrow$$ If Bob measures his spin along z, he certainly find $|\downarrow_z\rangle_B$

Similarly, if Alice finds $|\downarrow\rangle_A$, Bob's post measurement state is $|\uparrow_z\rangle_B$. The spins of Alice and Bob are anticorrelated along z. But now consider the following state

$$\hat{\rho}_{AB} = \frac{1}{2}|\uparrow_z\rangle_A\langle\uparrow_z| \otimes |\downarrow_z\rangle_B\langle\downarrow_z| + \frac{1}{2}|\downarrow_z\rangle_A\langle\downarrow_z| \otimes |\uparrow_z\rangle_B\langle\uparrow_z|$$

Here too, the marginals are completely mixed $\hat{\rho}_A = \hat{\rho}_B = \frac{1}{2}\hat{\mathbb{1}}$, so the invidual outcomes are random, but the measurement outcomes are correlated

$$P_{AB}(\uparrow_z, \downarrow_z) = P_A(\downarrow_z, \uparrow_z) = \frac{1}{2}, \quad P_{AB}(\uparrow_z, \uparrow_z) = P_{AB}(\downarrow_z, \downarrow_z) = 0 \Rightarrow$$ Measurement outcomes are anticorrelated along z

Moreover Alice can predict Bob's spin measurement along-z, based on her measurement.

$$\hat{\rho}_B|_{\uparrow_z^A} = \frac{{}_A\langle\uparrow_z|\hat{\rho}_{AB}|\uparrow_z\rangle_A}{P^A_{\uparrow z}} = |\downarrow_z\rangle_B\langle\downarrow_z|, \quad \hat{\rho}_B|_{\downarrow_z^A} = \frac{{}_A\langle\downarrow_z|\hat{\rho}_{AB}|\downarrow_z\rangle_A}{P^A_{\downarrow z}} = |\uparrow_z\rangle_B\langle\uparrow_z|$$

So, what's different about the entangled state? This state is <u>classically correlated</u>; it is a statistical mixture of possible correlation. The entangled state is a <u>coherent superposition</u> of possible correlations. Thus, for example, suppose Alice measures the spin along any other axis $\vec{n}$. $|\uparrow_{\vec{n}}\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle$, $|\downarrow_{\vec{n}}\rangle = \cos\frac{\theta}{2}|\downarrow_z\rangle - e^{-i\phi}\sin\frac{\theta}{2}|\uparrow_z\rangle$. For the singlet $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_{\vec{n}}\rangle|\downarrow_{\vec{n}}\rangle - |\downarrow_{\vec{n}}\rangle|\uparrow_{\vec{n}}\rangle)$

Thus, if Alice measures $\vec{n}\cdot\hat{\vec{\sigma}}$ and finds $|\uparrow_{\vec{n}}\rangle_A$ Bob's state is projected to $|\downarrow_{\vec{n}}\rangle$ and vice versa. The measurement outcomes are perfectly anti-correlated along any axis.

This is a special feature of entanglement, impossible with classical correlations.

Schmidt Decomposition

Consider the representation of a pure state $|\Psi\rangle_{AB} \in \mathcal{H}_{AB}$ w.r.t a product basis

$$\gamma^{AB}_{ij} = \langle ij|\Psi\rangle_{AB}; \quad |\Psi\rangle_{AB} = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B} \gamma^{AB}_{ij}|e_i\rangle_A \otimes |f_j\rangle_B$$

Elements of a $d_A \times d_B$ matrix (generally rectangular). Every such matrix can be decomposed according to a <u>singular value decomposition</u>.

$$\gamma = \mathbb{U}^T \mathbb{D} \mathbb{V},$$ where $\mathbb{U}, \mathbb{V}$ are unitary matrices of dimension $d_A \times d_A$ and $d_B \times d_B$ respectively, and $\mathbb{D}$ is a diagonal (rectangular) matrix, $\mathbb{D}_{\mu\nu} = \lambda_\mu \delta_{\mu\nu}$, where # of nonzero λ's $\leq \min(d_A, d_B)$.

The $\{\lambda_\mu\}$ are known as the "singular values" of the matrix.

Note $\quad \gamma^\dagger \gamma = V^\dagger \underbrace{D^\dagger D}_{\lambda_\mu^2 \delta_{\mu\nu}} V \qquad \gamma\gamma^\dagger = U^T D D^\dagger (U^T)^\dagger$

$\Rightarrow \{\lambda_\mu\} = \sqrt{\text{eigenvalues of } D^\dagger D}\ \}$

↖ real > 0

U, V define basis change transformations from $|e_i\rangle$ $|f_j\rangle$ to eigenvectors of $D^\dagger D$ and $DD^\dagger$

Thus $\gamma_{ij} = \sum_{\mu=1}^{\mu_{max}} U^T_{i\mu} \lambda_\mu V_{\mu j} = \sum_{\mu=1}^{\mu_{max}} \lambda_\mu U_{\mu i} V_{\mu j}$

$$\Rightarrow |\Psi\rangle_{AB} = \sum_{\mu=1}^{\mu_{max}} \lambda_\mu \underbrace{\left(\sum_{i=1}^{d_A} U_{\mu i} |e_i\rangle_A\right)}_{\equiv |u_\mu\rangle_A} \otimes \underbrace{\left(\sum_{j=1}^{d_B} V_{\mu j} |f_j\rangle_B\right)}_{|v_\mu\rangle_B}$$

$$\Rightarrow \boxed{|\Psi\rangle_{AB} = \sum_{\mu=1}^{\mu_{max}} \lambda_\mu |u_\mu\rangle_A \otimes |v_\mu\rangle_B}$$

This is know as the <u>Schmidt decomposition</u>

- The Schmidt decomposition differs from the expansion in arbitrary basis in that it involves the sum over one index.

- the bases $\{|u_\mu\rangle_A\}$ and $\{|v_\mu\rangle_B\}$ are known as the <u>Schmidt bases</u>. They are functions of the state $|\Psi\rangle_{AB}$

- the coefficients $\{\lambda_\mu\}$ are known as the <u>Schmidt numbers</u> and μ_{max} = # of non-zero $\{\lambda_\mu\}$ is the <u>Schmidt rank</u>.

- If the Schmidt rank = 1, the state is <u>separable</u>

$$|\Psi\rangle_{AB} = |u_1\rangle_A \otimes |v_1\rangle_B$$

- If the Schmidt rank > 1, the state is entangled.

- The degree of entanglement is typically measured by the <u>entropy of the singular values</u> ^{2}, i.e., the entropy of $p_\mu = |\langle u_\mu v_\mu | \Psi \rangle_{AB}|^2 = \lambda_\mu^2$

$$E = \sum_{\mu=1}^{\mu_{max}} -\lambda_\mu^2 \log_2 \lambda_\mu^2$$

We typically measure the log in base-2 so entropy in "bits"

$$0 \leq E \leq \log_2 d \qquad (d = \min(d_A, d_B))$$

(0: separable; $\log_2 d$: maximally entangled)

Eg. Two qubits $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \Rightarrow$ Schmid decomposita

$$|\Psi\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle|\downarrow\rangle + \frac{1}{\sqrt{2}}(i|\downarrow\rangle)(i|\uparrow\rangle))$$

$\lambda^2 = (\frac{1}{2}, \frac{1}{2})$ max entropy $E = 1$ bit

<u>Physical meaning of entropy as measure of entanglement</u>

Consider marginal density operators:

$$\hat{\rho}^A = Tr_B(|\Psi\rangle_{AB}\langle\Psi|) = Tr_B\left(\sum_{\mu\mu'} \lambda_\mu \lambda_{\mu'} |u_\mu\rangle_A\langle u_{\mu'}| \otimes |v_\mu\rangle_B\langle v_{\mu'}|\right)$$

$$\Rightarrow \hat{\rho}^A = \sum_{\mu=1}^{\mu_{max}} \lambda_\mu^2 |u_\mu\rangle_A\langle u_\mu|, \text{ Similarly, } \hat{\rho}^B = \sum_{\mu=1}^{\mu_{max}} \lambda_\mu^2 |v_\mu\rangle_B\langle v_\mu|$$

The entanglement is related to the purity of $\hat{\rho}^A$, $\hat{\rho}^B$

The Schmidt bases are (up to a phase) the eigenvectors of the marginal states

- The Von-Neumann entropy of the marginals $\hat{\rho}^A$ and $\hat{\rho}^B$ are equal and determined the entanglement of $|\Psi\rangle_{AB}$

$$E_{|\Psi\rangle_{AB}} = S(\hat{\rho}_A) = S(\hat{\rho}_B) = -\mathrm{Tr}(\hat{\rho}_A \log_2 \hat{\rho}_A)$$

This is one of the most profound results of quantum theory. Even though we have maximal information about the joint system, it is in a pure state, $|\Psi\rangle_{AB}$, if it is entangled we have an incomplete description of the parts taken alone; they are in mixed state $\hat{\rho}^A$, $\hat{\rho}^B$. The missing information is in the quantum correlations between A + B. The entropy of the state represents ~~our~~ the missing information about the state. When the state is pure, its entropy is zero. When it is maximally mixed, its entrop is $\log d$. If $|\Psi\rangle_{AB}$ is maximally entangled, and $d_A = d_B = d$, then $\hat{\rho}_A$ and $\hat{\rho}_B$ are maximally mixed.

The generation of entanglement

Entanglement corresponds to the quantum correlation between degrees of freedom. Thus, to create entanglement, one must have interactions between these degrees of freedom. A Hamiltonian which acts separately on the two degrees of freedom is said to be "separable"

$$\hat{H}_{BA} = \hat{H}_A \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{H}_B$$

Such as Hamiltonian has separable (product-state) eigenstates, $|\Psi_{nm}\rangle_{AB} = |e_n\rangle_A |f_m\rangle_B$. The unitary evolution acts "locally" on each subsystem

$$\hat{U}_{AB} = e^{-i\hat{H}_{AB}t} = e^{-i(\hat{H}_A \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{H}_B)t} = \underbrace{e^{-i\hat{H}_A t}}_{\hat{U}_A} \otimes \underbrace{e^{-i\hat{H}_B t}}_{\hat{U}_B}$$

When there are interactions between the subsystems, the Hamiltonian is not seperable

$$\hat{H}_{AB} \neq \hat{H}_A \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{H}_B \quad \text{e.g.} \quad \hat{H}_{AB} = \hat{H}_A \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{H}_B + \hat{H}_A^{int} \otimes \hat{H}_B^{int}$$

$$\Rightarrow \hat{U}_{AB} = e^{-i\hat{H}_{AB}t} \neq \hat{U}_A \otimes \hat{U}_B$$

Such a unitary map is said to be an "entangling unitary" in that

$$\hat{U}_{AB} |\psi\rangle_A \otimes |\phi\rangle_B \quad \text{is an entangled state}$$

The eigenstates of the interacting Hamiltonian are entangled

$$|\Psi_{nm}\rangle_{AB} \neq |e_n\rangle_A \otimes |f_m\rangle_B$$

<u>Entanglement and the choice of subsystem "coordinate."</u>

In classical physics, we know that the choice of coordinate is critical in simplying the problem to allow for a solution. For example, for a coupled set of oscillators, we can write the solution instantly in terms of "normal mode" coordinates. A famous example is "two-body" problem, e.g. the Hydrogen atom. We can consider two subsystems, the electron and the nucleus (proton)

$$\hat{H}_{Hyd} = \hat{H}_{eN} = \frac{\vec{P}_e^2}{2m_e} + \frac{\vec{P}_N^2}{2m_N} - \frac{e^2}{|\vec{r}_e - \vec{r}_N|} \neq \hat{H}_e \otimes \hat{\mathbb{1}}_N + \hat{\mathbb{1}}_e \otimes \hat{H}_N$$

This Hamiltonian is <u>not separable</u> in electron and proton degrees of freedom. However, we know that we can separate <u>center-of-mass</u> and <u>relative</u> coords.

$$\hat{H}_{Hyd} = \hat{H}_{cr} = \frac{\vec{P}_c^2}{2M} + \frac{\vec{P}_r^2}{2\mu} - \frac{e}{r} = \hat{H}_c \otimes \hat{\mathbb{1}}_r + \hat{\mathbb{1}}_c \otimes \hat{H}_r$$

The eigenfunctions of the hydrogen atom are product states in center of mass and relative coordinates

$$|\Psi\rangle_{Hydrogen} = |\vec{k}\rangle_c \otimes |n,l,m\rangle_{rel} \neq |\psi\rangle_e \otimes |\phi\rangle_N$$

↑ center of mass free particle plane wave

↖ Hydrogenic wave function for the relative motion

Thus, we see that the choice of subsystem can determine whether or not the system is entangled or not.

The Hydrogen Hamiltonian is entangling in electron + proton coordinates but separable in relative and center-of-mass coordinates.

Note on change of basis/coordinates and entanglement

There are two different notions of a "basis change" that can be confusing, but are relevant to the question of entanglement.

- Change of "local basis". Given a multipartite system, e.g. bipartite, with Hilbert space $\mathcal{H}_{AB} = h_A \otimes h_B$, the entanglement of the state is unchanged by a local change of basis on h_A and h_B.

$$\text{Given } |\Psi\rangle_{AB} = \sum \gamma_{ij} |e_i\rangle_A \otimes |f_j\rangle_B = \sum \delta_{ij} |d_i\rangle_A \otimes |g_j\rangle_B$$

where $|d_i\rangle_A = \hat{U}_A |e_i\rangle_A$, $|g_j\rangle_B = \hat{U}_B |f_j\rangle_B$ (local basis change)

$\gamma_{ij} = {}_A\langle e_i| \otimes {}_B\langle f_j|\Psi\rangle_{AB}$ + $\delta_{ij} = {}_A\langle d_i| \otimes {}_B\langle g_j|\Psi\rangle_{AB}$ have the same Schmidt decomposition

- Change of subsystem decomposition

The change of coordinates $(x_A, x_B) \Rightarrow \left(\frac{x_A - x_B}{\sqrt{2}}, \frac{x_A + x_B}{\sqrt{2}}\right)$ is a basis change of the classical degrees of freedom. This is not a local basis change on the subsystem describing the motion of particle A + B.