Physics 581: Lecture 6
Entangled States of the Quantized Electromagnetic Field

How do we describe entangled states of the quantized E/M field, and how are the produced?

The simplest example is the entanglement of two photons.

- Polarization entanglement: Consider two photons, each in a specified mode, whose state is specified by the polarization of the photon. The Hilbert is then specifying two qubits

$$\mathcal{H}_{AB} = h_{1/2}^{A} \otimes h_{1/2}^{B} \qquad \text{(modes A, B)}$$

Standard basis for polarization state on Poincaré sphere $\{|H\rangle = \hat{a}_H^\dagger |0\rangle, |V\rangle = \hat{a}_V^\dagger |0\rangle\}$

General Pure State in the Standard Basis

$$|\Psi\rangle_{AB} = \gamma_{HH}|HH\rangle + \gamma_{HV}|HV\rangle + \gamma_{VH}|VH\rangle + \gamma_{VV}|VV\rangle$$

Entanglement: Schmidt decomposition of γ : $\quad S = \mathrm{Tr}_B\left(|\Psi\rangle_{AB}\langle\Psi|\right)$

- Momentum entanglement : Each photon has momentum as well as spin (polarization). We consider a bipartite Hilbert space

$$\mathcal{H}_{AB} = h_{\mathcal{L}_2} \otimes h_{\mathcal{L}_2}$$

where $|\psi\rangle \in h_{\mathcal{L}_2}$, $|\psi\rangle = \int d^3\vec{k}\, \psi(\vec{k})\, |\vec{k}\rangle$, $\int d^3k\, |\psi(\vec{k})|^2 = 1$

$$|\vec{k}\rangle = \hat{a}^\dagger_{\vec{k},\vec{\epsilon}}|0\rangle = |1\rangle_{\vec{k},\vec{\epsilon}}$$

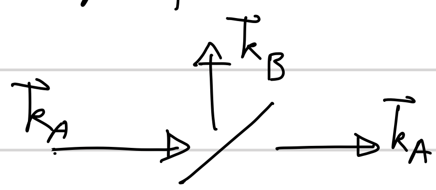
Generall state $|\Psi\rangle_{AB} = \int d^3\vec{k}\, d^3\vec{k}'\, \Psi(\vec{k},\vec{k}')\, \hat{a}^\dagger_{\vec{k}\vec{\epsilon}_A} \hat{a}^\dagger_{\vec{k}'\vec{\epsilon}_B} |0\rangle$

Need infinite dimensional generalization of Schmidt

$$\Psi(\vec{k},\vec{k}') = \sum_{\mu=1}^{\mu_{max}} \lambda_\mu\, u_\mu(\vec{k})\, v_\mu(\vec{k}) \quad : \quad \text{Entangled } \mu_{max} > 1$$

Mode entanglement vs. Particle entanglement

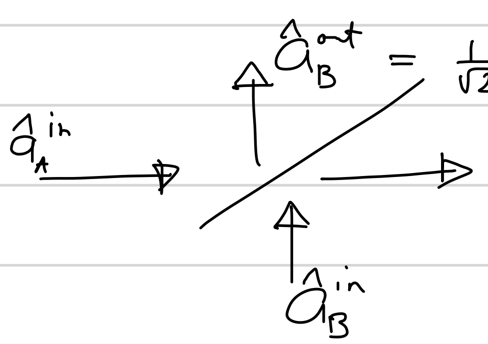
Consider a single photon incident on a 50-50 beam splitter



The momentum space wave function becomes a superposition of plane waves

$$|\vec{k}_A\rangle \Rightarrow \frac{1}{\sqrt{2}}\left(|\vec{k}_A\rangle + |\vec{k}_B\rangle\right)$$

One wouldn't generally call such a state entangled. Or would you? As written, we have used "first quantization" notation, where the state is that of a single photon. Suppose, however, we use "second quantization." The field amplitude scattering matrix is the transformation on the mode annihilation operators : $\hat{a}_k^{out} = \hat{S}\hat{a}_k^{in}\hat{S}^\dagger = \sum_{k'} \Lambda_{kk'}\hat{a}_{k'}$



$$\hat{a}_B^{out} = \frac{1}{\sqrt{2}}\left(\hat{a}_A^{in} - \hat{a}_B^{in}\right)$$

$$\hat{a}_A^{out} = \frac{1}{\sqrt{2}}\left(\hat{a}_A^{in} + \hat{a}_B^{in}\right)$$

(50-50 beam splitter with certain phases)

Let $|\Psi\rangle_{in} = \hat{a}_A^{in\,\dagger}|0\rangle_A \otimes |0\rangle_B = |1,0\rangle$

$$\Rightarrow |\Psi\rangle_{out} = \hat{S}|\Psi\rangle_{in} = \hat{S}\hat{a}_A^{in\,\dagger}\hat{S}^\dagger|0\rangle_A \otimes |0\rangle_B$$

$$= \frac{1}{\sqrt{2}}\left(\hat{a}_A^\dagger + \hat{a}_B^\dagger\right)|0\rangle_A \otimes |0\rangle_B = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right)$$

This states now looks entangled. In fact it is with respect to the modes not the particles. When we treat the quantized field we can manipulate the particle degrees of freedom (discrete variables) or the wave degrees of freedom (continuous variables). From the point of view of the CV, the beam splitter creates entanglement!

We can see this another way. Consider the quadrature operators,

$$\hat{X} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}, \quad \hat{P} = \frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}}$$

The beam splitter transformation

$$\hat{S}\hat{X}_A\hat{S}^\dagger = \frac{\hat{X}_A+\hat{X}_B}{\sqrt{2}}, \qquad \hat{S}\hat{X}_B\hat{S}^\dagger = \frac{\hat{X}_A-\hat{X}_B}{\sqrt{2}}$$

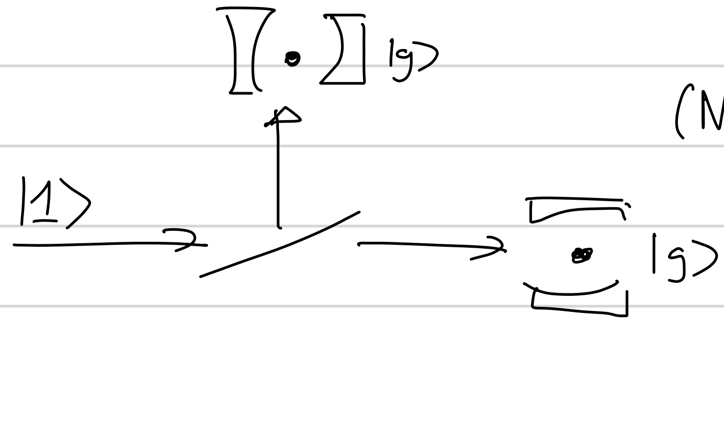
That is, the beam splitter transforms between A/B coordinates are "center-of-mass"/"relative" coordinates. We know and separable state in one subsystem decomposition looks entangled in the other.

The fact that the beam-splitter produces entanglement is surprising because the action is by linear optics and thus involves no interactions between the photons. Entanglement requires interaction between the degrees of freedom. However, in the perspective we are currently considering, the degrees of freedom are the modes, not the particles. The Hamiltonian that generates the beam splitter transformation is of the form

$$\hat{H} = K\hat{a}_B^\dagger\hat{a}_A + K^*\hat{a}_A^\dagger\hat{a}_B$$

This is not separable in the modes.

To convince ourselves further that the state produced by sending a single photon to a beam splitter is entangled, consider the following Gedanken experiment



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A single photon wave packet is sent to a 50-50 beam splitter. At each output port is an atom cavity in its ground state $|g\rangle$. The cavity coupling is such that if a single photon passes through the cavity $|1\rangle\otimes|g\rangle \rightarrow |0\rangle\otimes|e\rangle$ according to the Jaynes-Cummings interaction.

Thus, the total state of the system: Two modes of field $\otimes$ Two atoms transforms according to:

$$|1_A, 0_B\rangle \otimes |g,g\rangle \Rightarrow \frac{1}{\sqrt{2}}\left(|1_A, 0_B\rangle + |0_A, 1_B\rangle\right)|g,g\rangle$$

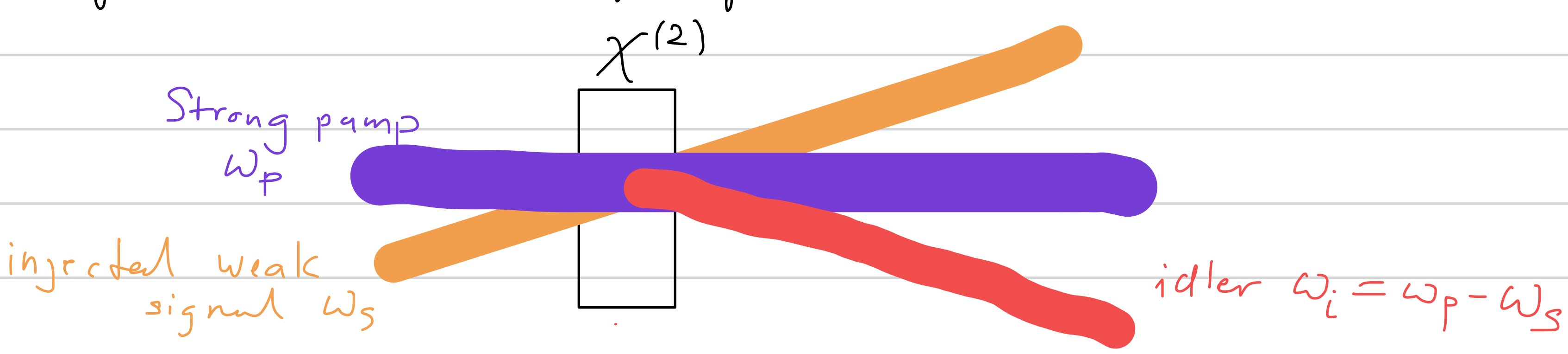
$$\Rightarrow |0_A 0_B\rangle \otimes \frac{1}{\sqrt{2}}\left(|e\rangle|g\rangle + |g\rangle|e\rangle\right)$$

The entanglement in the field was transfered to the atoms. No one would dispute that the two-atom state $\frac{1}{\sqrt{2}}\left(|e\rangle|g\rangle + |g\rangle|e\rangle\right)$ is entangled.

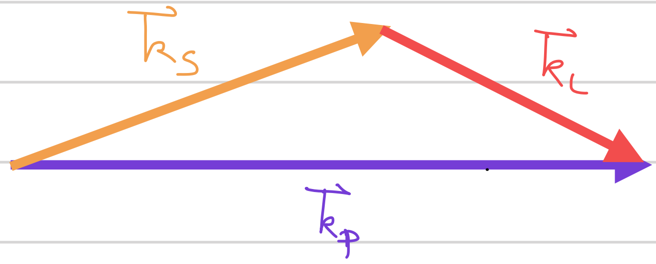
Note: In order to generated an entangled state using a beam splitter, we require nonclassical night ($P(\alpha)<0$). A product of coherent states remains separable.

Generating Entangled States of Light: Spontaneous Parametric Downconversion

The workhorse for generating entangled states of light is the nonlinear optical process of parametric downconversion, ~~typically~~ via three way mixing in a $\chi^{(2)}$ crystal, whereby a pump photon is annihilated and a pair of photons (the "signal" and "idler") are created. We briefly considered this in Lecture 3 in the context of pametric amplification, whereby a signal field (frequency ω_s, wave vector $\vec{k}_s$) in the presence of a pump (frequency ω_p, wave vector $\vec{k}_p$) is amplified while simultaneously generating an "idler" field (frequency ω_i, wave vector $\vec{k}_i$).

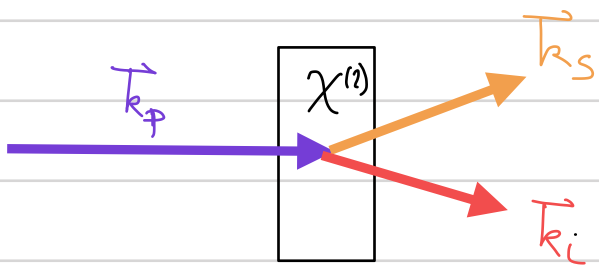


This process is strong only when the "phase-matching" condition is satisfied

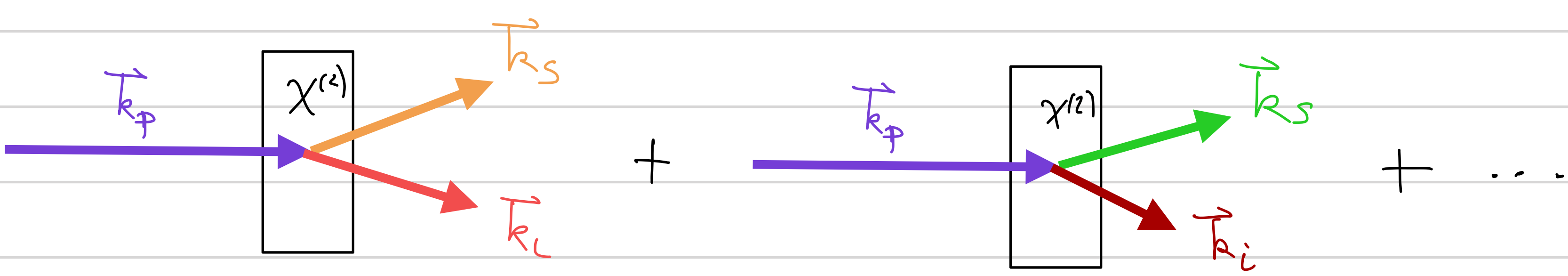


$$\vec{k}_p = \vec{k}_s + \vec{k}_i$$
$$\omega_p = \omega_s + \omega_i$$

The amplification process can be considered as a "stimulated" emission process of a photon into the signal mode, stimulated by pump + signal. The process will also occur spontaneously when there are no injected signal (or idler) photons. We call this process spontaneous parametric downconversion (SPD).



Because this is a spontaneous process, there are an (uncountably) infinite number of possible down conversion process, all of which satisfy the phase matching condition. Because they are indistinguishable, they occur in superposition. The result is a momentum entangled state



The pattern of spontaneously emitted photon pairs depends on the phase matching condition $\omega_p = \omega_i + \omega_s$, $\vec{k}_p = \vec{k}_s + \vec{k}_i$

$$\Rightarrow \omega_p n_p(\omega_p) = \omega_s n_s(\omega_s)\cos\theta_s + \omega_i n_i(\omega_i)\cos\theta_i \quad \text{(Longitudinal component)}$$
$$0 = \omega_s n_s(\omega_s)\sin\theta_s + \omega_i n_i(\omega_i)\sin\theta_i \quad \text{(transverse component)}$$

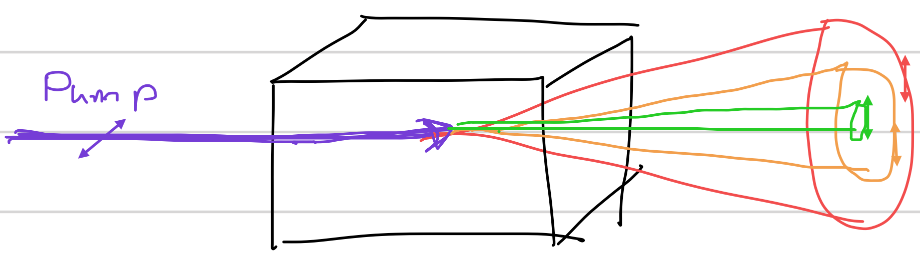
To satisfy the phase matching conditions in three-wave mixing, we must employ birefringence. This comes in two forms

"Type I" phase matching: $n_p(\omega) = n_e(\omega)$ "extraordinary" polarization
$n_s(\omega) = n_i(\omega) = n_o(\omega)$ "ordinary polarization"

"Type II" phase matching: $n_p(\omega) = n_e(\omega)$ "extraordinary" polarization
$n_s(\omega) = n_e(\omega)$ "extra ordinary", $n_i(\omega) = n_o(\omega)$ "ordinary polarization"

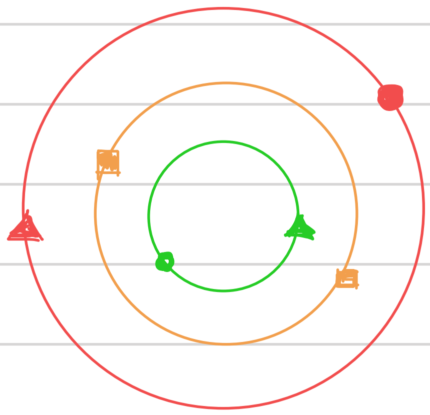
The type of phase matching exployed affects the entangled state produced and the pattern of spontaneously emitted photons

In type-I phase matching, signal and idler photons have the same polarization. The different processes are distinguished by their momenta (and color)



Parametric down conversion rainbow on cones.

Conjugate pairs on opposite side of cones such that $\omega_p = \omega_s + \omega_i$



Circles, Squares, and triangles represent different conjugate pairs. Typically apperatures are used at the output face to filter out a subset of the possible processes.

Formally, the state of system is generated by the Hamiltonian we saw in lecture 3 for three-way mixing (in the nondepleted pump limit)

$$\int dt'\, \hat{H}(t') = K \int d^3\vec{k}_s\, d^3\vec{k}_i\, \underbrace{\delta(\omega_s+\omega_i-\omega_p)\, \delta^{(3)}(\vec{k}_s+\vec{k}_i-\vec{k}_p)}_{\text{idealized phase matching}}\, \hat{a}^\dagger_{\vec{k}_s}\, \hat{a}^\dagger_{\vec{k}_i} + h.c.$$

depends on $\chi^{(2)}$, pump intensity

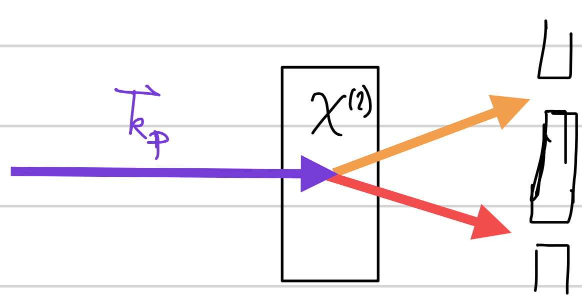
For type-I phase matching in one crystal, we drop the polarization index, since both signal & idler have the same polarization ("ördinary").

The state $|\Psi\rangle = e^{-\frac{i}{\hbar}\int dt'\, H(t')}|0\rangle = |0\rangle - i\hat{H}L/c\,|0\rangle + \cdots$

Two-photon component:

$$|\Psi_2\rangle = \int d^3\vec{k}_s\, d^3\vec{k}_i\, \underbrace{\Psi(\vec{k}_s,\vec{k}_i)}_{\text{two-photon wave function}}\, |1\rangle_{\vec{k}_s} \otimes |1\rangle_{\vec{k}_i} \;:$$

Often, we discuss this type of state is terms of "energy-time" entanglement. Suppose we place pinholes that narrow the range of directions of propagation of signal + idler. There can still be a broad-spectrum of frequencies that can pass the pinholes.



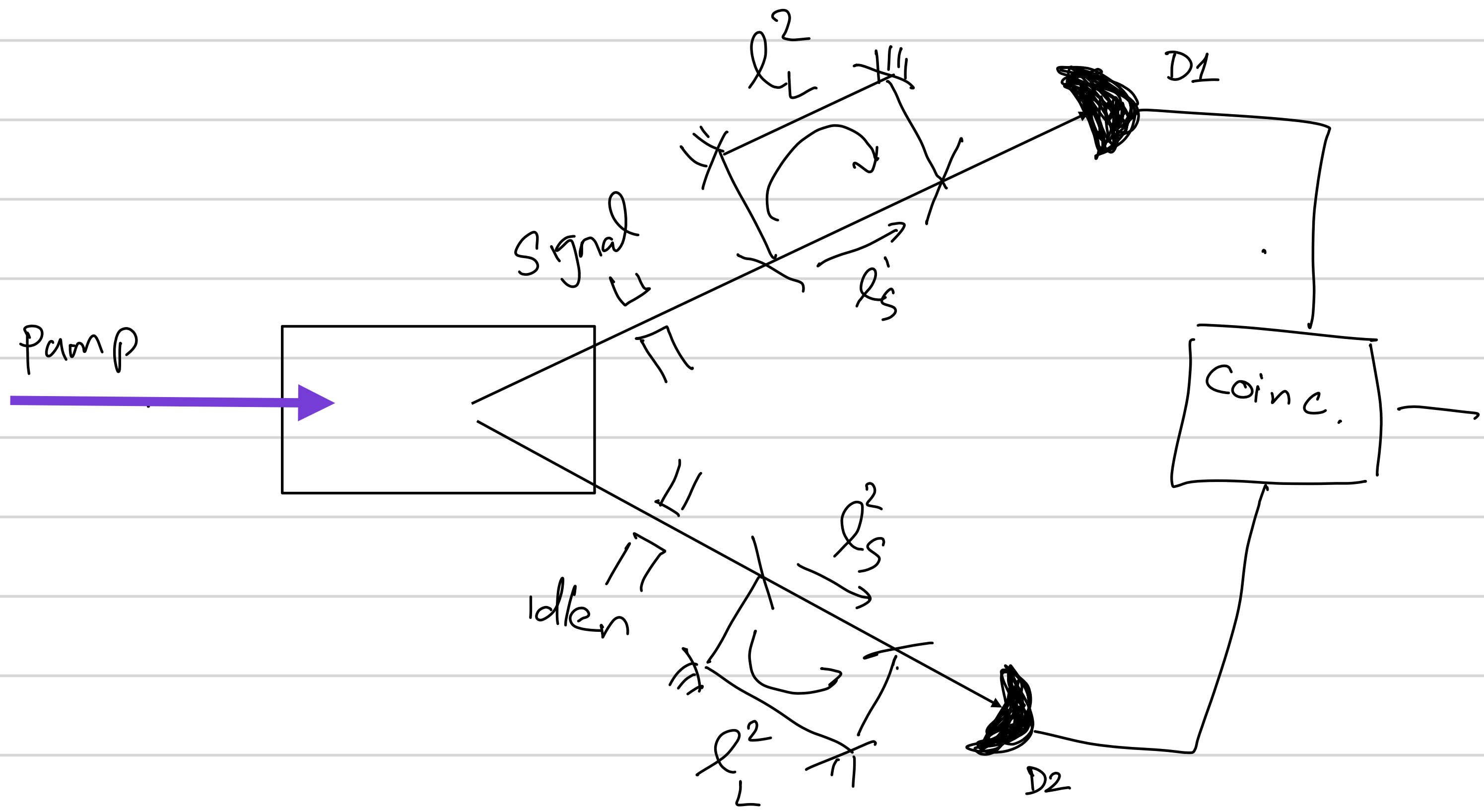
The state that passes the pinhole can be written in simplified form

$$|\Psi_2\rangle = \int d\omega_s \int d\omega_i\, \delta(\omega_p-(\omega_s+\omega_i))\, \underset{\uparrow}{\tilde{A}(\omega_s)}\, |1\rangle_{\omega_s}\, |1\rangle_{\omega_i} = \int dt_s\, dt_i\, e^{-i\omega_p t_i}\, A(t_s-t_i)\, |1\rangle_{t_s} |1\rangle_{t_i}$$

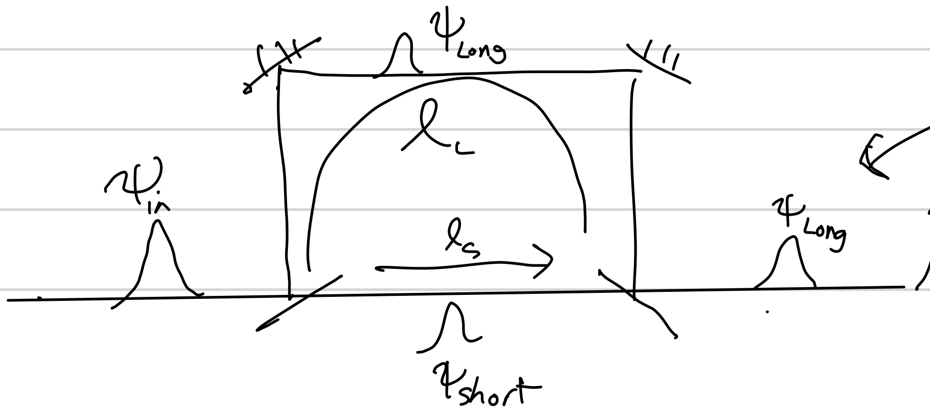
spectral function depending on pinhole

The two photons have frequency that add to ω_p, but the frequency of ω_s (and thus $\omega_i = \omega_p - \omega_s$) is in superposition. In the dual "time bin", variables, the photons are born nearly simultaneously, separated in time $\Delta(t_s - t_i) \sim \frac{1}{\Delta(\omega_s - \omega_i)}$, but their birthday is completely uncertain to the degree that we have a CW pump and infinitely long crystal, $\Delta\left(\frac{t_s+t_i}{2}\right) \sim \frac{1}{\Delta\omega_p}$

The indistinguishability of the birthday of the twin photons implies the possibility of interference between different birthdays. Consider the following interferometer

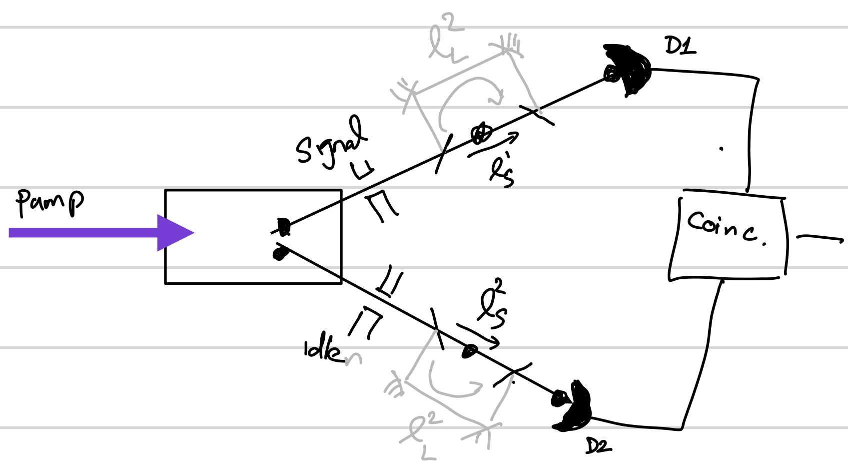


The signal and idler photons are each sent into independent interferometers the are <u>unbalanced</u> with a short path l_s and a long path l_L. The coherence length of each photon $\frac{c}{\Delta\omega_s}$, $\frac{c}{\Delta\omega_i}$ is chosen to be much shorter that the path difference $\Delta l = l_L - l_s$. In that case, the "singles counts" in each individual detector, D1 and D2 with show no interference; neither the signal photon nor the idler photon will "interfere with themselves." That is the short path and the long path are distinguishable for the wideband (short time) single photon wave packets

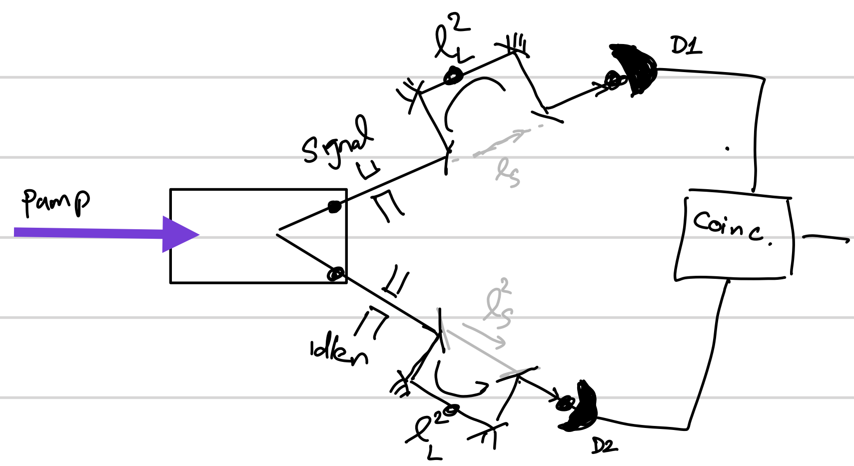


Distinguishable wave packets: ψ_{short} does not overlap with ψ_{Long}. It arrives earlier.

However, in coincidence, there are two indistinguishable histories that can lead to the same outcome: Both photons take the short path, and both photons take the long path. For these two histories to be indistinguishable, they need to correspond to two possible different birthdays, where the two photons that take the long path were born earlier than the two photons that took the short path. Then these two histories correspond to two possible simultaneous coincidences:



Both photons take short path



Both photons take long path

For given signal and idler frequencies ω_s and ω_i, the probability amplitude for joint detection is proportional to

$$\Psi_{coinc} \propto e^{i\frac{\omega_s}{c}\ell_s^1} e^{i\frac{\omega_i}{c}\ell_s^2} + e^{i\frac{\omega_s}{c}\ell_L^1} e^{i\frac{\omega_i}{c}\ell_L^2}$$

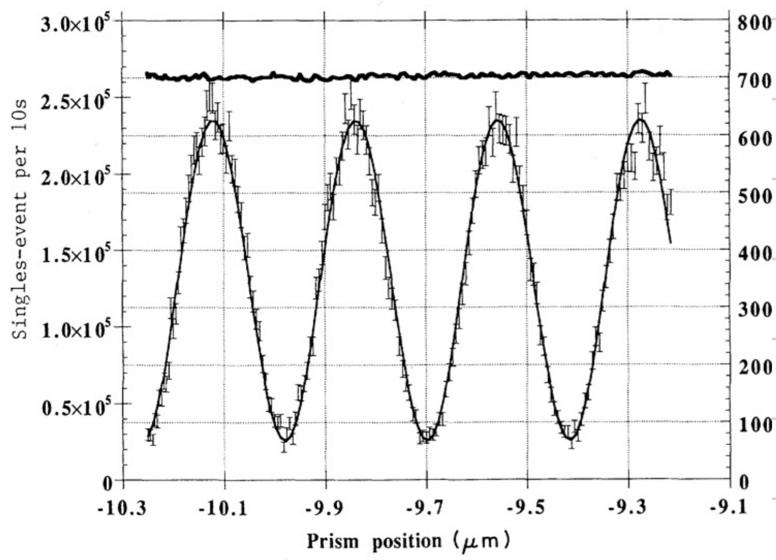
$$\Rightarrow P_{coinc} = \frac{1}{2}\left[1 + \cos\left(\frac{\omega_s}{c}\Delta\ell_1 + \frac{\omega_i}{c}\Delta\ell_2\right)\right] \quad \text{where } \Delta\ell = \ell_L - \ell_s$$

$$= \frac{1}{2}\left[1 + \cos\left(\omega_p\left(\frac{\Delta t_1 + \Delta t_2}{2}\right) + (\omega_s - \omega_i)\left(\frac{\Delta t_1 - \Delta t_2}{2}\right)\right)\right] \qquad \Delta t = \frac{\Delta\ell}{c}$$

Here we assumed that $\omega_s + \omega_i = \omega_p$ exactly. This must then be integrated over the amplitudes for ω_s, consistent with phase matching.

If $\Delta(t_1 - t_2) << \frac{1}{\Delta(\omega_s - \omega_i)}$: The difference in the imbalance is chosen to be small compared to $\frac{1}{\Delta\omega}$

$$\Rightarrow P_{coinc} \propto 1 + \cos\left(\omega_p \frac{(\Delta t_1 + \Delta t_2)}{c}\right) \; : \; \text{Two-photon interference}$$



Data from Kwiat, Steinberg, & Chiao
Phys. Rev. A 47 R2472 (1993).

Singles counts show no interference;
the coincidence counts show two-photon interference representing energy-time entanglement

Spatial mode entanglement: Orbital angular momentum

We have seen that in SPD with a single crystal, we can create either momentum entangled (including time-energy) or polarization-entangled photon pairs depending on the type of phase matching used. Using "first quantization" notation

$$\text{Type-I}: |\Psi\rangle = \int_{\vec{k}_s \vec{k}_i} d^3\vec{k}_s\, d^3\vec{k}_i\, \Psi(\vec{k}_s, \vec{k}_i) |\vec{k}_s\rangle \otimes |\vec{k}_i\rangle \qquad \text{(both polarization same)}$$

The two-photon momentum space wave function

$$\Psi(\vec{k}_s, \vec{k}_i) = \int d\vec{k}_p\, \underbrace{\tilde{E}(\vec{k}_p)}_{\text{pump beam}}\, \underbrace{\xi(\vec{k}_p - (\vec{k}_s + \vec{k}_i))}_{\text{phase matching depending of crystal}}\, \delta(\omega_p - (\omega_s + \omega_i))$$

Let us choose two frequencies ω_s & ω_i which add to ω_p, and directions signal and idler $\vec{k}_s$ and $\vec{k}_i$ that are phase-matched near $\vec{k}_p$ (paraxial)

Define $\vec{k} = \vec{q} + k_z \vec{e}_z$ $\quad \vec{q}$ = transverse component $\quad \Rightarrow k_z = \sqrt{k - q^2}$

$k = \frac{\omega}{c} n(\omega)$

$\Rightarrow$ For fixed frequency, two-photon wave function depends only of $\vec{q}_s, \vec{q}_i$ in the paraxial approximation

$$\Rightarrow |\Psi\rangle = \int d^2\vec{q}_1\, d^2\vec{q}_2\, \Phi(\vec{q}_1, \vec{q}_2)\, |\vec{q}_1\rangle |\vec{q}_2\rangle$$

In is useful to decompose this into a basis of "tranverse spatial modes" that represent complete complete set. Consider the "Laguerre-Gaussian" modes

$$\mathcal{U}_{\ell,p}(r_\perp, \phi) = \sqrt{\frac{w^2 p!}{2\pi(p+|\ell|)!}} \left(\frac{w r_\perp}{2}\right)^{|\ell|} e^{-\frac{w^2 r_\perp^2}{2}} (i)^{\ell} (-1)^p \mathcal{L}_p^{|\ell|}\left(\frac{w^2 r_\perp^2}{2}\right) e^{i\ell\phi}$$

these are eigenstates of orbital angular moment (OAM) ℓ, along z-axis

These are complete $\Rightarrow$ $\sum_{\ell_1, p_1} \sum_{\ell_2 p_2} |\ell_1 p_1\rangle\langle \ell_1 p_1| \otimes |\ell_2 p_2\rangle\langle \ell_2 p_2| = \hat{\mathbb{1}}_1 \otimes \hat{\mathbb{1}}_2$ for two-photon spatial modes

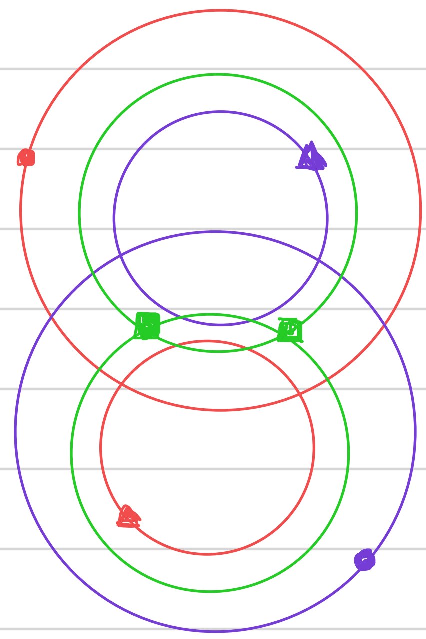
$$\Rightarrow |\Psi\rangle = \sum_{\ell_1 \ell_2 p_1 p_2} C_{\ell_1 \ell_2, p_1 p_2} |\ell_1 p_1\rangle \otimes |\ell_2 p_2\rangle$$

where $C_{\ell_1 p_1, \ell_2 p_2} = \int dq_1\, dq_2\, \Phi(\vec{q}_1, \vec{q}_2)\, \underbrace{\tilde{\mathcal{U}}^*_{\ell_1 p_2}(\vec{q}_1)\, \tilde{\mathcal{U}}_{\ell_1, p_2}(\vec{q}_2)}_{\text{Laguerre-Gaussian in momentum sp}}$

OAM is entangled in SPDC.

Type II Phase Matching

In type-II phase matching, the signal and idler photons have orthogonal linear polarizations, which we will denote H (Horizontal) + V (Vertical), and on nonconcentric cones. By appropriate choice of geometry, we can obtain the following emission pattern



H cones ("signal")

V cones ("idler")

The two-photon wave function takes the form

$$|\Psi_2\rangle = \int d^3\vec{k}_s\, d^3\vec{k}_i\; \delta(\omega_p - (\omega_s + \omega_i))\, \delta(\vec{k}_p - (\vec{k}_s + \vec{k}_i))\, \hat{a}^\dagger_{\vec{k}_s H}\, \hat{a}^\dagger_{\vec{k}_i V}\, |0\rangle$$

For a two special directions, $\vec{k}_s = \vec{k}_i$ (where green circles intersect) the signal + idler photons are indistinguishable w.r.t. to $\vec{k}$. In that situation, the signal and idler become entangled in polarization

$$\frac{1}{\sqrt{2}}\left(|H_1 V_2\rangle + e^{i\alpha}|V_1, H_2\rangle\right) \qquad \text{(Next Page)}$$

phase depends on crystal

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{\vec{k}_1 H}\, \hat{a}^\dagger_{\vec{k}_2 V} + e^{i\alpha}\, \hat{a}^\dagger_{\vec{k}_1 V}\, \hat{a}^\dagger_{\vec{k}_2 H}\right)|0\rangle$$

In first quantized notation,

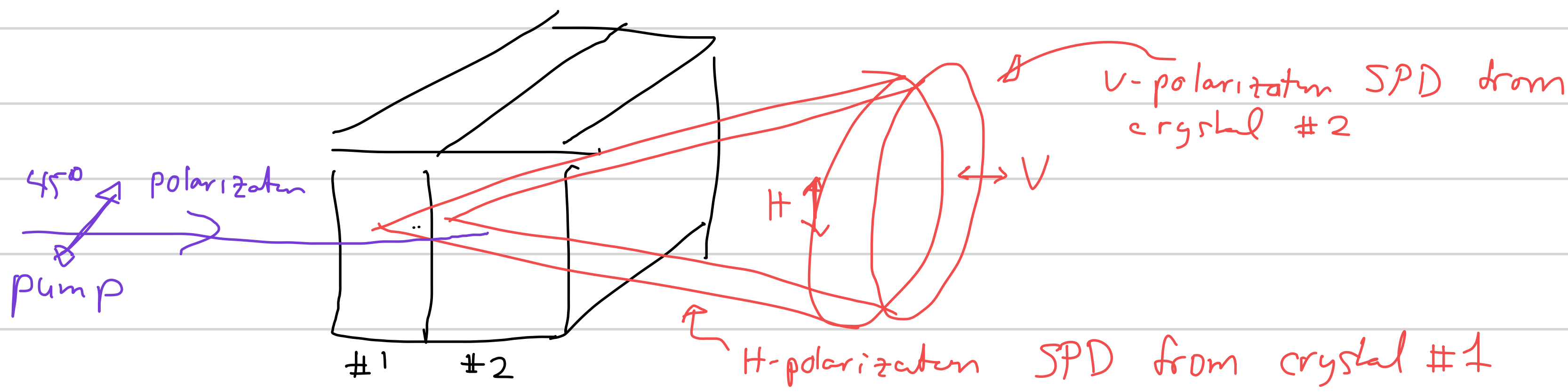
$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left[(|\vec{k}_1\rangle \otimes |H\rangle) \otimes (|\vec{k}_2\rangle \otimes |V\rangle) + (|\vec{k}_1\rangle \otimes |V\rangle) \otimes (|\vec{k}_2\rangle \otimes |H\rangle)\right]$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left[|H,V\rangle + e^{i\alpha}|V,H\rangle\right]}_{\text{Two photon polarization-entangled state}} \otimes |\vec{k}_1, \vec{k}_2\rangle$$

Because the momentum of the photon does not distinguish the polarization, the momentum degrees of freedom are separable from the polarization thus when we trace over the momentum and only measure the polarization, we arrive at a maximally entangled polarization state.

Hyperentanglement

Two-crystal hyper entangled state (see Kwiat et al, PRA 60 R773 (1999)



The crystals are oriented such that type-I SPD occurs in each crystal with H-polarization downconverted photons produced in #1 when the pump is V-polarized and V-polarized downconverted photons produced in #2 when the pump is H-polarized. By orienting the pump at 45°, and for crystals thin enough the two possible down conversion processes are indistinguishable. In that case, we add the probability amplitudes of each of these process and the result is

$$|\Psi_2\rangle = \int d^3k_s\, dk_i\ \delta(\omega_s+\omega_i-\omega_p)\,\delta(\vec{k}_s+\vec{k}_i-\vec{k}_p)\left(\frac{a^{\dagger}_{\vec{k}_s,H}\,a^{\dagger}_{\vec{k}_i,H}+e^{i\alpha}\,a^{\dagger}_{\vec{k}_s,V}\,a^{\dagger}_{\vec{k}_i,V}}{\sqrt{2}}\right)|0\rangle$$

or in first quantized notation

$$\boxed{|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left(|HH\rangle + e^{i\alpha}|VV\rangle\right)\otimes\int d\omega_s\, |\omega_s\rangle\otimes|\omega_p-\omega_s\rangle\otimes\int d\vec{q}_s\, d\vec{q}_i\ f(\vec{q}_i,\vec{q}_s)\,|\vec{q}_i\rangle\otimes|\vec{q}_s\rangle}$$

Mode Entanglement

The entanglement we have studied so far is "particle entangled," i.e., entanglement associated with the degree of freedom of the photons (particles) — polarization and/or momentum. What about "mode entanglement," i.e. entanglement between the wave degrees of freedom (quadratures).

Consider again, the spontaneous parametric downconversion process. We know that this produces squeezed vacuum. For a single mode degenerate OPA, the interaction generates the squeezing operator acting on the vacuum

$$\hat{S}(r)|0\rangle = e^{\frac{r}{2}(\hat{a}^{\dagger 2}-\hat{a}^2)}|0\rangle = e^{\frac{\tanh r}{2}\hat{a}^{\dagger 2}} e^{-\ln(\cosh r)(\hat{a}^\dagger\hat{a}+\frac{1}{2})} e^{-\frac{\tanh r}{2}\hat{a}^2}|0\rangle$$

(See R. Traux, PRD 31 1988 (1985).)

$$= \frac{1}{\sqrt{\cosh r}} e^{\frac{\tanh r}{2}\hat{a}^{\dagger 2}}|0\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tanh r}{2}\right)^n (\hat{a}^\dagger)^{2n}|0\rangle$$

$$\Rightarrow |0_r\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{\sqrt{2n!}}{2^n n!}(\tanh r)^n|2n\rangle \underset{r\text{ small}}{\simeq} |0\rangle + \frac{r}{\sqrt{2}}|2\rangle + \dots$$

$$\hat{S}^\dagger\hat{X}\hat{S} = e^{-r}\hat{X}, \quad \hat{S}^\dagger\hat{P}\hat{S} = e^{+r}\hat{P}, \quad \hat{X} = \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}, \quad \hat{P} = \frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}$$

Consider the two-mode (nondegenerate) OPO, studied in homework. For special choice of pump phase, the two-mode squeezing operator

$$\hat{S}_{AB}(r) = e^{r(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})} = e^{\tanh r\,\hat{a}^\dagger\hat{b}^\dagger}\left(\frac{1}{\cosh r}\right)^{\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}+1} e^{-\tanh r\,\hat{a}\hat{b}}$$

The two-mode squeezed vacuum

$$|0,0\rangle_r = \hat{S}_{AB}|0\rangle\otimes|0\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n|n\rangle_A\otimes|n\rangle_B$$

The two-modes are entangled. This is often refered to as "twin beams," because the two modes have perfect number correlations. This is sometimes called "number squeezing" because the intensity difference between the two modes is below "shot noise".

Note: The representation of the two-mode squeezed state is, as written, the Schmidt decomposition. The Schmidt basis is the Fock basis! (this is an example where the Schmidt basis is the same for a class of states). The Schmidt coefficients are

$$\lambda_n = \frac{(\tanh(r))^n}{\cosh(r)} \Rightarrow \text{Probability distribution } P_n = \lambda_n^2 = \frac{\sinh^{2n} r}{(\cosh^2 r)^{n+1}}$$

The marginal states are $\hat{\rho}_{A,B} = \sum_{n=0}^{\infty} P_n |n\rangle_{A,B}\langle n| = \sum_{n=0}^{\infty} \frac{(\sinh^2 r)^n}{(1+\sinh^2 r)^{n+1}} |n\rangle_{A,B}\langle n|$

This is a thermal state with $\langle n\rangle = \sinh^2 r$, $\langle n+1\rangle = \cosh^2 r$ $\Rightarrow \hat{\rho}_{A,B} = \sum_{n=0}^{\infty} \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} |n\rangle_{A,B}\langle n|$

The mode entanglement is the Von Neumann entropy of these states

$$E = \mathrm{tr}(-\hat{\rho}\log\hat{\rho}) = -\sum_n P_n\, n \log\langle n\rangle + \sum_n P_n (n+1)\log(1+\langle n\rangle)$$

$$= -\langle n\rangle \log\langle n\rangle + \langle n+1\rangle \log(\langle n+1\rangle) = -\sinh^2 r \log(\sinh^2 r) + \cosh^2 r \log(\cosh^2 r)$$

Note, the amount of "mode entanglement" goes to infinity as $r \to \infty$ for the infinite dimensional Hilbert space. $E \to 2r \log e$

The thermal marginal states are Gaussian States, i.e. their Wigner functions are Gaussian. This gives us indication that the two-mode squeezed state has a Gaussian Wigner function.

The two mode Wigner function

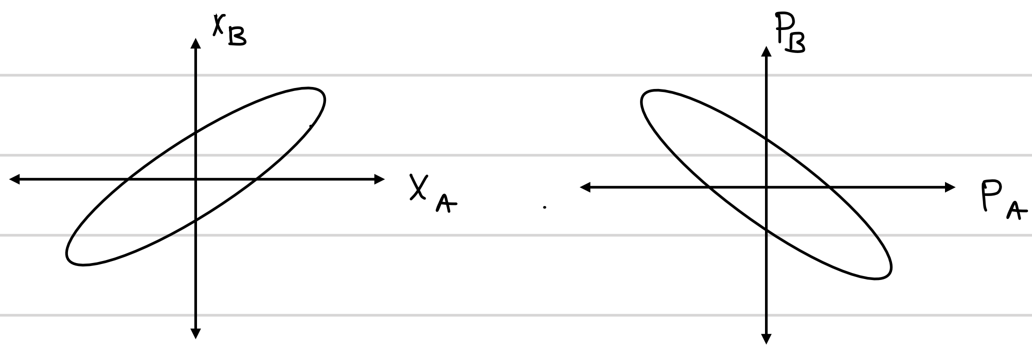
$$W(\alpha,\beta) = \int \frac{d^2\alpha'}{\pi^2}\int \frac{d^2\beta'}{\pi^2} \langle 0,0|\, \hat{S}^\dagger_{AB}(r)\, \hat{D}_A(\alpha')\, \hat{D}_B(\beta')\, \hat{S}_{AB}(r)\,|0,0\rangle$$

$$= \frac{4}{\pi^2}\, e^{-2\cosh 2r\,(|\alpha|^2+|\beta|^2) + 2\sinh 2r\,(\alpha\beta+\alpha^*\beta^*)}$$

$$\Rightarrow W(X_A, X_B, P_A, P_B) \propto e^{-\sigma^2 (X_A+X_B)^2}\, e^{-\frac{(X_A-X_B)^2}{\sigma^2}}\, e^{-\frac{(P_A+P_B)^2}{\sigma^2}}\, e^{-\sigma^2(P_A-P_B)^2}$$

where $\sigma^2 = e^{-2r}$

$$= |\Psi(X_A,X_B)|^2\, |\tilde{\Psi}(P_A,P_B)|^2$$

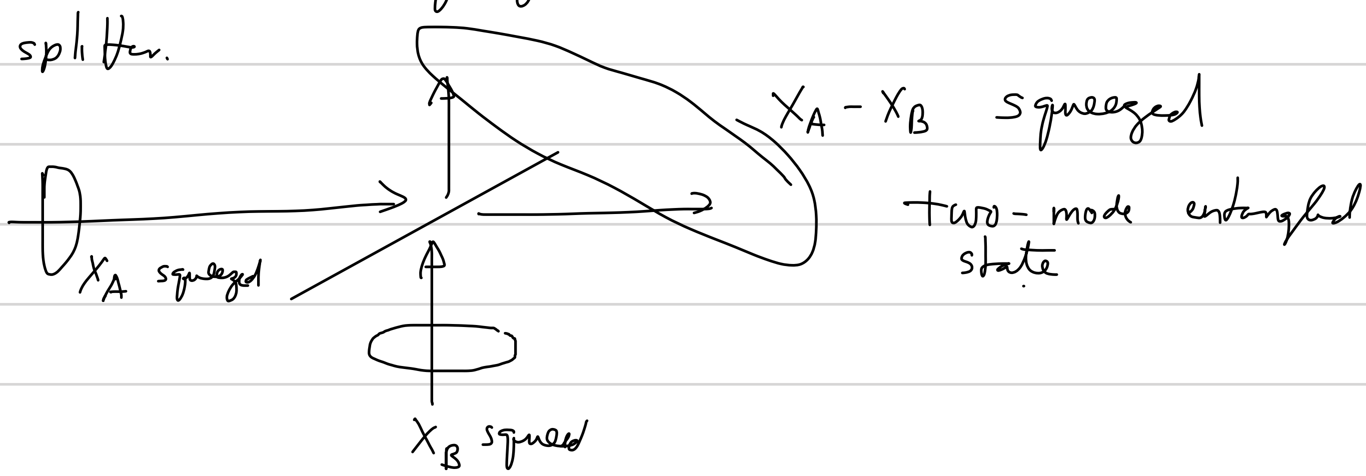


Thus the difference $\hat{X}_A - \hat{X}_B$ (the sum $\hat{P}_A + \hat{P}_A$) is squeezed
sum $\hat{X}_A + \hat{X}_B$ (difference $\hat{P}_A - \hat{P}_B$) is antisqueezed

<u>Bogoliubov:</u> $\hat{S}^\dagger_{AB}\, \hat{a}_{B/A}\, \hat{S}_{AB} = \cosh r\, \hat{a}_{A/B} + \sinh r\, \hat{a}^\dagger_{B/A}$

$$\hat{S}^\dagger_{AB}(\hat{X}_A \pm \hat{X}_B)\hat{S}_{AB} = (\hat{X}_A \pm \hat{X}_B)e^{\pm r}, \qquad \hat{S}^\dagger_{AB}(\hat{P}_A \pm \hat{P}_B)\hat{S}_{AB} = (\hat{P}_A \pm \hat{P}_B)e^{\mp r}$$

<u>Note:</u> Making NOPO is more challenging than a single mode degenerate OPO. However, one can create the same entangle state by just sending to independently created single-mode squeezed vacuum states onto a 50-50 lossless beam splitter.



The EPR Paradox and Bell's Inequalities

In the limit of infinite squeezing

$$W(X_A, X_B, P_A, P_B) \Rightarrow \delta(X_A - X_B)\, \delta(P_A + P_B)$$

$$\Rightarrow |0,0\rangle_{r\to\infty} = \int dX\, |X\rangle_A \otimes |X\rangle_B = \int \frac{dP}{2\pi}\, |P\rangle_A\, |-P\rangle_B = \sum_n |n\rangle_A |n\rangle_B \quad \text{(maximally entangled)}$$

This is a variant of the famous state considered by Einstein-Podolsky-Rosen (EPR)

$$\Psi_{EPR}(X_A, X_B) = \delta(X_A - X_B - X_0); \qquad \tilde{\Psi}_{EPR}(P_A, P_B) = e^{-iP_B X_0}\, \delta(P_A + P_B)$$

$$(\hat{X}_A - \hat{X}_B)|EPR\rangle = X_0 |EPR\rangle, \qquad (\hat{P}_A + \hat{P}_B)|EPR\rangle = 0$$

EPR used this state to "prove" the quantum mechanics was "incomplete," i.e., that the randomness that we observed is due to incomplete information stored locally in "hidden variables." The argument is straightforward and compelling, involving the interplay of two hallmark differences between classical and quantum physics — incompatible (noncommuting) variables and entanglement.

"Element of reality" If without in any way disturbing a system we can predict with certainty the value of a physical quanitity, then there exists an "element of a reality" associated with this quantity.

The EPR state is a simultaneous eigenstate of $\hat{X}_A + \hat{X}_B$ and $\hat{P}_A - \hat{P}_B$, two comutting observables, and thus an allowed state of quantum mechanics. It is not, however, a simultaneous eigenstate of $\hat{X}_A$ and $\hat{P}_A$ nor $\hat{X}_B$ and $\hat{P}_B$, which are incompatible (noncomutting) variables. On the other hand, because of the entanglement measuring $\hat{X}_A$ we instantly learn the value of $\hat{X}_B$. Since subsystem A and B are space-like separated (by X_0), by Einstein-locality, a measurement of A can in no way affect subsystem B. Thus, EPR call X_B and "element of reality." In other words, the X_B of X_B we learned was objectively locally real. Similarly, because of entanglement, if we measure $\hat{P}_A$ we learn the value of $\hat{P}_B$.

Thus by Einstein locality, P_B is also an "element of reality". Therefore it follows that both X_B and P_B are "elements of reality" and thus quantum mechanics is incomplete — i.e., both X and P "exist," and when we measure them we just learn what their value is. The randomness can be completed if we had access to the "local hidden variable."

The EPR paradox is simple and beautiful and cuts to the heart of what makes no sense about quantum mechanics. Of course there is one flaw; we cannot measured X_A and P_A at the same time, so we cannot deduce what would happen if measure X_A & P_A at the same time by consider separate (incompatible) experiments – one where we measure X_A and one where we measure P_A. This was Bohr's response, but it did not settle anything between Bohr & Einstein.

All of this changed with the analysis of John Bell in 1964, where he showed that the existence of "local elements of reality" also known as "local hidden variables" had a measureable consequence. That is any local hidden variable theory would lead to measurements that satisfy an inequalities – such inequalities are known as "Bell's inequalities."

(Local) Hidden Variables

In a hidden variable theory, there are parameters, generically called λ, whose values determine any measurement outcome, i.e., the measurement outcome is deterministic according to some underlying theory. But, because the value of λ is "hidden" from us, the outcome appear random. The idea of locality introduced by Bell is inspired by EPR – that the value that we read in one subsystem cannot depend on the measurement obtained in another subsystem if the two subsystems are space-like separated.

Thus, for a biparite system, Alice measures her subsystem with measurement setting "a", and Bob " his " " " " "b". The values the find are

$$A(a,\lambda) \quad \text{and} \quad B(b,\lambda)$$

The key to locality is that A is independent of b and vice versa.
Note: the hidden variable is shared be A & B. For example, in a dynamical model, A can be correlated with B, but only if at some earlier time A & B shared some information, e.g. some initial condition of the system.

Because λ is a hidden variable, we attribute only probabilities to their values $p(\lambda)$. Then the expected values are

$$\langle A(a) \rangle_{LHV} = \int d\lambda \, p(\lambda) \, A(a,\lambda)$$

$$\langle B(b) \rangle_{LHV} = \int d\lambda \, p(\lambda) \, B(b,\lambda)$$

$$\langle A(a) B(b) \rangle_{LHV} = \int d\lambda \, p(\lambda) \, A(a,\lambda) \, B(a,\lambda)$$

Ironically, the state considered by EPR does not violated local realism. This is because there exists a classical local probability distribution that predicts the probability of seeing a given outcome — The Wigner Function. The two-mode squeezed state's Wigner function is everywhere positive. (See Chapt. 21 of J. Bell "Speakable and Unspeakable in Quantum Mechanics").

David Bohm had considered a variation of the EPR based on entangled spin-½ particles (qubits) — a finite dimension case which does not succumb to the same problem of the EPR state

Consider the spin singlet

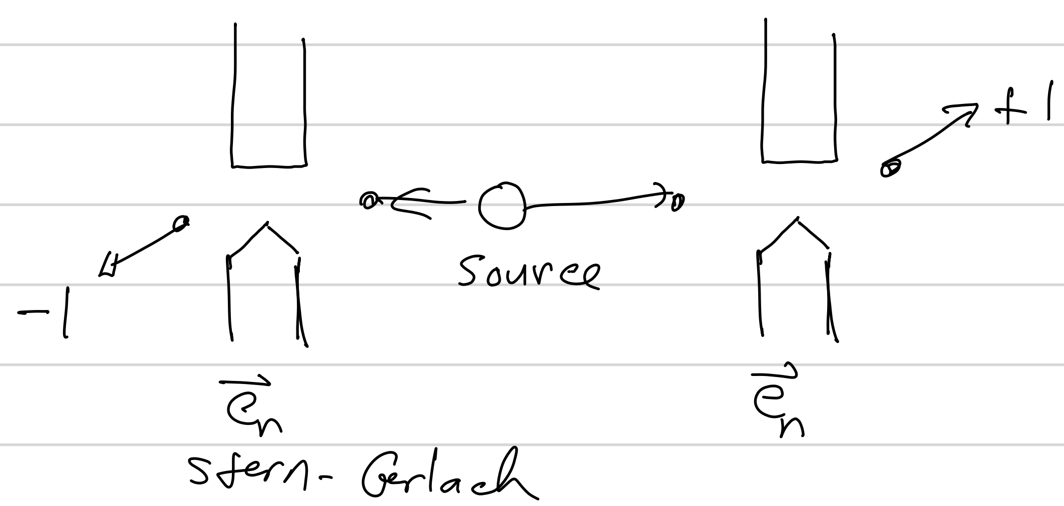
$$|\Psi_{sing}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right)$$

The singlet has total angular momentum $S=0$ and thus it is an eigenstate of spin projection along any axis $\vec{e}_n$ with eigenvalue 0.

$$\vec{e}_n\cdot\left(\hat{\vec{\sigma}}_A + \hat{\vec{\sigma}}_B\right)|\Psi_{sing}\rangle = 0$$

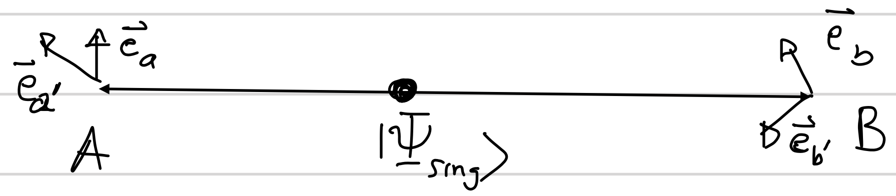
or $$\left(\vec{e}_n\cdot\hat{\vec{\sigma}}_A\right)|\Psi_{sing}\rangle = -\left(\vec{e}_n\cdot\hat{\vec{\sigma}}_B\right)|\Psi_{sing}\rangle$$

The results are perfectly anticorrelated. Thus measuring a spin-projection of A instantly tells of the projection all the same direction for B. The value is either "up" $\vec{e}_n\cdot\hat{\vec{\sigma}} = +1$ or down $\vec{e}_n\cdot\hat{\vec{\sigma}} = -1$



Bell defined an inequality that must be satisfied by a LHV model if the state showed such exact anticorrelation. In any physical realization we never have a perfect spin singlet, so it is necessary to relax this condition, and consider arbitrary states and how a LHV constrains correlations. This was first done by Clauser, Horne, Shimony, & Holt (CHSH) in, PRL 23 880 (1969)

Consider the following Gedanken Experiment



Alice chooses to measure her spin along $\vec{e}_a$ or $\vec{e}_{a'}$; Bob along $\vec{e}_b$ or $\vec{e}_{b'}$. The values then measure find the "local values" that are "elements of reality" Then consider the following observable

$$\hat{S} = \hat{\sigma}_{\vec{a}} \otimes (\hat{\sigma}_{\vec{b}} - \hat{\sigma}_{\vec{b}'}) + \hat{\sigma}_{\vec{a}'} \otimes (\hat{\sigma}_{\vec{b}} + \hat{\sigma}_{\vec{b}'}) \qquad \text{where} \qquad \hat{\sigma}_n = \vec{e}_n \cdot \hat{\vec{\sigma}}$$

In a LHV mode

$$S_{LHV}(\lambda) = A(\vec{a},\lambda)\underbrace{[B(\vec{b},\lambda) - B(\vec{b}',\lambda)]}_{0 \text{ or } \pm 2} + A(\vec{a}',\lambda)\underbrace{[B(\vec{b},\lambda) + B(\vec{b}',\lambda)]}_{\pm 2 \text{ or } 0}$$

since $B = \pm 1 \quad \forall \vec{b}, \vec{b}', \lambda$

$$S_{LHV}(\lambda) = \pm 2 \text{ for all } \lambda \Rightarrow |S_{LHV}(\lambda)| = 2 \quad \forall \lambda$$

$$\Rightarrow \langle |S_{LHV}| \rangle = 2 \Rightarrow |\langle S_{LHV} \rangle| \leq \langle |S_{LHV}| \rangle = 2$$

$$\boxed{|\langle S \rangle|_{LHV} = \left| \langle \sigma_{\vec{a}} \otimes \sigma_{\vec{b}} \rangle + \langle \sigma_{\vec{a}'} \otimes \sigma_{\vec{b}} \rangle + \langle \sigma_{\vec{a}'} \otimes \sigma_{\vec{b}'} \rangle - \langle \sigma_{\vec{a}} \otimes \sigma_{\vec{b}'} \rangle \right|_{LHV} \leq 2}$$

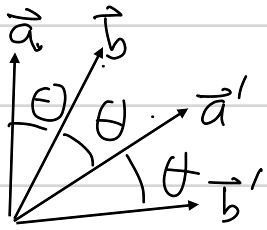
This is known as the CHSH (Clauser-Horn-Shimony-Holtz) form of the Bell's inequality (Clauser et al., PRL 23 880 (1969)). It is a generalization of the original Bell's inequality (J. Bell, Physics 1 195 (1964)), which assumed a perfect singlet state.

Quantum mechanically $\langle \sigma_{\vec{n}} \otimes \sigma_{\vec{n}'} \rangle = -\vec{e}_n \cdot \vec{e}_{n'}$

$$\Rightarrow |\langle S \rangle|_{QM} = |\vec{a}\cdot\vec{b}' - (\vec{a}\cdot\vec{b} + \vec{a}'\cdot\vec{b} + \vec{a}'\cdot\vec{b}')| .$$

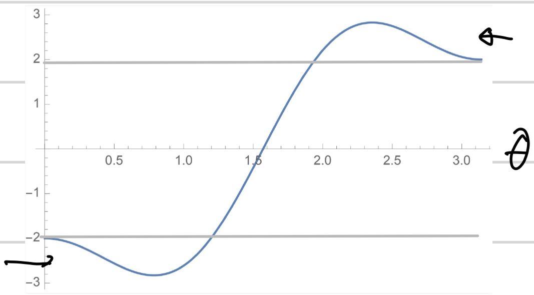
Can this violate Bell's inequalities?

Consider the following choices of $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$. Choose in plane



$$\Rightarrow \langle S \rangle_{QM} = \cos 3\theta - 3\cos\theta \qquad |\langle S \rangle_{QM}|_{max} = 2\sqrt{2} \text{ for } \theta = 45^\circ$$

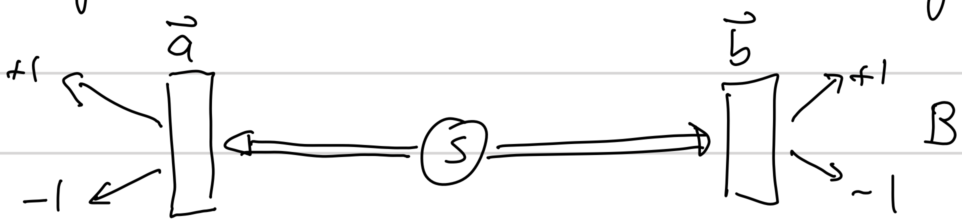
> 2 Violates CSHS-Bell!



Violates LHV theory!

Experimental Test

A generic test of Bell's inequalities involves a source of entangled particles that sends subsystems to remotely located Alice and Bob who independently measure in local bases set by direction $\vec{a}$ and $\vec{b}$



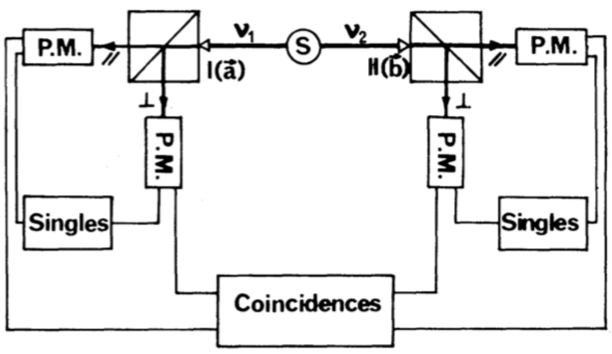
the experimental determined expected value

$$\langle (\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) \rangle = \langle (|\uparrow_a\rangle\langle\uparrow_a| - |\downarrow_a\rangle\langle\downarrow_a|) \otimes (|\uparrow_b\rangle\langle\uparrow_b| - |\downarrow_b\rangle\langle\downarrow_b|) \rangle$$

$$= P_{\uparrow_a\uparrow_b} + P_{\downarrow_a\downarrow_a} - P_{\downarrow_a\uparrow_b} - P_{\uparrow_a\downarrow_b} = \frac{N_{\uparrow_a\uparrow_a} + N_{\downarrow_a\downarrow_a} - N_{\downarrow_a\uparrow_b} - N_{\uparrow_a\downarrow_b}}{N_{\uparrow_a\uparrow_a} + N_{\downarrow_a\downarrow_a} + N_{\downarrow_a\uparrow_b} + N_{\uparrow_a\downarrow_b}}$$

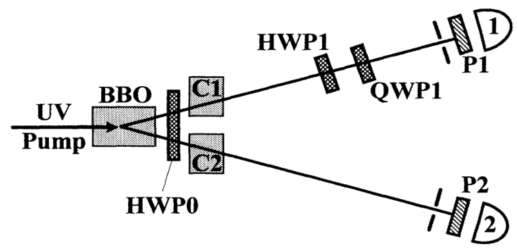
Normalized this way accounts only for coincidence events and takes only a <u>sub ensemble</u> when we don't have perfect detectors

The seminal experiment was performed in the group of Alain Aspect A. Aspect, P. Grangier, & G. Roger PRL <u>49</u>, 91 (1981), building on the pioneering experiments of John Clauser PRL <u>28</u>, 938 (1972).



Polarization entangled photons were created in atomic vapor in a "cascade" emission discussed last semester

Modern experiments to test Bell's inequalities with entangled photons are done with SPDC polarization correlated pairs



PG Kwiat et al. PRL <u>75</u> 4337 (1995). Using type-II downconversion; they found violation of Bell's inequalities by over 100 standard deviations!

There are subtle loopholes in the experimental tests:

– "Locality" "Freedom-of-choice" "Fair-sampling"

Only in 2015 have these all been closed simultaneously. Three experiments:

- Hansen et al., Nature <u>526</u>, 682 (2015) (Spins in Solids)
- Giustin et al., Phys. Rev. Lett <u>115</u>, 250401 (2015) (Photons)
- Shalm et al., Phys. Rev. Lett. <u>115</u>, 250402 (2015) (Photons)

Quantum Teleportation

While violations of Bell's inequalities show that the data we see is inconsistent with a locally realist description of nature, in recent decades we have learned that entanglement is a resource for performing quantum information processing tasks. A beautiful example is "quantum teleperation"

The idea of quantum teleportation is simple. Alice wants to send a quantum state $|\psi\rangle_A$ to Bob. She doesn't have access at the time to a "quantum channel" through which she can set $|\psi\rangle_A$ to Bob. Moreover, with a single copy of $|\psi\rangle_A$ she cannot determine all the probabilities that define $|\psi\rangle_A$ in order to call Bob on the phone to have him make the state himself; she can only obtain $\log_2 d$ bits of information – e.g. up or down for d-dimensional system.
The idea is easiest to see for the case of qubits. We define the "Bell basis" (really might be called the Bohm basis)

$$|\Psi\rangle_\pm = \frac{1}{\sqrt{2}}\left(|0\rangle_A |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B\right)$$

$$|\Phi\rangle_\pm = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B\right)$$

These four states are characterized by two "quantum numbers." they are the simultaneous eigenvectors of $\Big\{ \underbrace{\hat{\sigma}_x^A \otimes \hat{\sigma}_x^B}_{\substack{\text{"parity bit"} \\ a=\pm 1}} , \underbrace{\hat{\sigma}_z^A \otimes \hat{\sigma}_z^B}_{\substack{\text{"phase bit"} \\ b=\pm 1}} \Big\}$

These operators have eigenvalues ± 1, thus, the Bell basis is defined by two bits $|a, b\rangle$.

$$|++\rangle = |\Phi_+\rangle, \quad |+,-\rangle = |\Phi_-\rangle, \quad |-,+\rangle = |\Psi_+\rangle, \quad |-,-\rangle = |\Psi_-\rangle$$

$$|0\rangle_A|0\rangle_B = \frac{1}{\sqrt{2}}\left(|++\rangle + |+-\rangle\right), \quad |1\rangle_A|1\rangle_B = \frac{1}{\sqrt{2}}\left(|++\rangle - |--\rangle\right), \quad |0\rangle_A|1\rangle_B = \frac{1}{\sqrt{2}}\left(|-,+\rangle + |-,-\rangle\right)$$

$$|1\rangle_A|0\rangle_B = \frac{1}{\sqrt{2}}\left(|-+\rangle - |--\rangle\right)$$

Consider thus, the situation in which Alice is handed an unknown (pure state) by Charlie $|\psi\rangle_C$ that she wants to teleport to B. They share an entangled state $|\Phi_+\rangle = |++\rangle$

The joint state of all three qubits

$$|\Psi\rangle_{CAB} = |\psi\rangle_C \otimes |\Phi_+\rangle_{AB} = (\alpha|0\rangle_C + \beta|1\rangle_C) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

$$= \frac{\alpha}{\sqrt{2}} |0\rangle_C \otimes |0\rangle_A \otimes |0\rangle_B + \frac{\beta}{\sqrt{2}} |1\rangle_C \otimes |1\rangle_A \otimes |1\rangle_B$$

$$+ \frac{\alpha}{\sqrt{2}} |0\rangle_C \otimes |1\rangle_A \otimes |1\rangle_B + \frac{\beta}{\sqrt{2}} |1\rangle_C \otimes |0\rangle_A \otimes |0\rangle_B$$

$$= |\Phi_+\rangle_{CA} \otimes (\alpha|0\rangle_A + \beta|1\rangle_B) + |\Phi_-\rangle \otimes (\alpha|0\rangle_B - \beta|1\rangle_B)$$

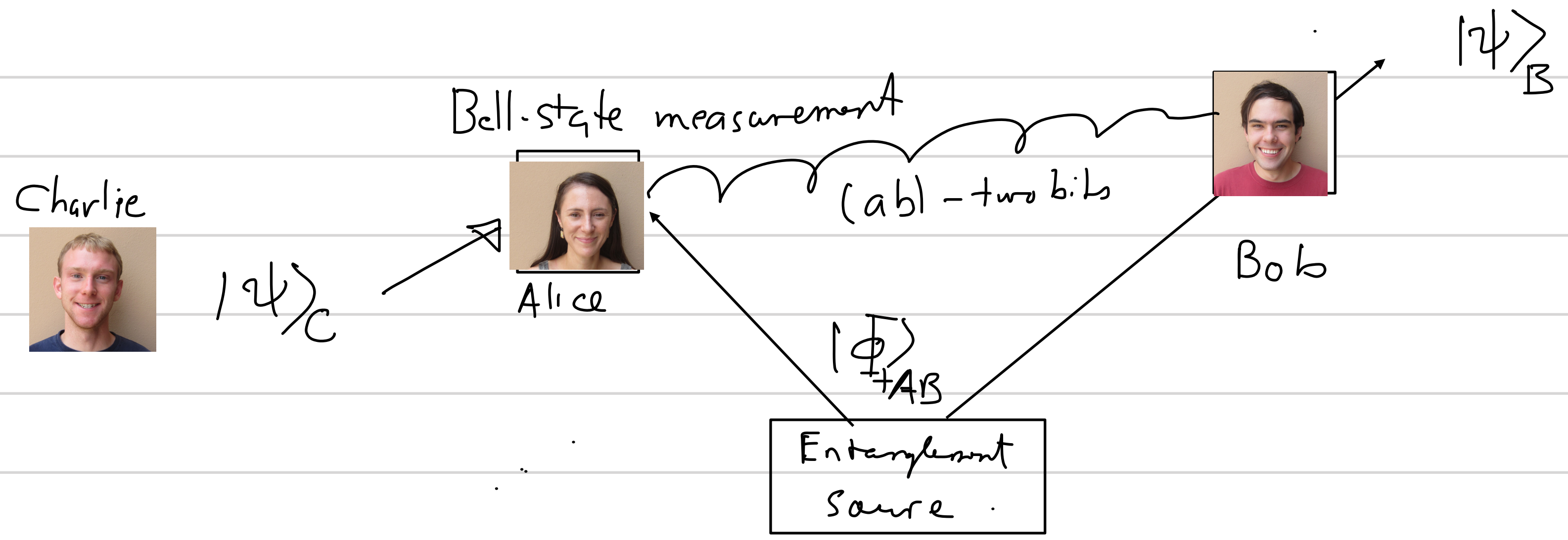
$$+ |\Psi_+\rangle_{CA} \otimes (\alpha|1\rangle_B + \beta|0\rangle_B) + |\Psi_-\rangle_{CA} \otimes (\alpha|1\rangle_B - \beta|0\rangle_B)$$

$$\Rightarrow |\Psi\rangle_{CAB} = |\Phi_+\rangle_{CA} \otimes |\psi\rangle_B + |\Phi_-\rangle_{CA} \otimes (\hat{\sigma}_z|\psi\rangle_B) + |\Psi_+\rangle_{CA} \otimes (\hat{\sigma}_x|\psi\rangle_B) + |\Psi_-\rangle_{CA} \otimes (\hat{\sigma}_x\hat{\sigma}_z|\psi\rangle_B)$$

$$\boxed{|\psi\rangle_C |\Phi_+\rangle_{AB} = \sum_{a,b} |a,b\rangle_{CA} \otimes (\hat{\sigma}_z)^{\left(\frac{1-a}{2}\right)} (\hat{\sigma}_x)^{\left(\frac{1-b}{2}\right)} |\psi\rangle_B}$$

The protocol for quantum teleportation is thus clear
- Alice performs of Bell state measurement on the joint state C & A
- She get two bits a & b and knows, thus that Bob's state is $(\hat{\sigma}_z)^{\frac{1-a}{2}} (\hat{\sigma}_x)^{\left(\frac{1-b}{2}\right)} |\psi\rangle$
- She calls Bob on the phone (or text's him the two bits)
- Bob then applies $(\hat{\sigma}_z)^{\frac{1-a}{2}} (\hat{\sigma}_x)^{1-b}$ to his qubit that was sent as part of the original Bell pair
- Because $\hat{\sigma}_i^2 = \mathbb{1}$ the result is $|\psi\rangle_B$

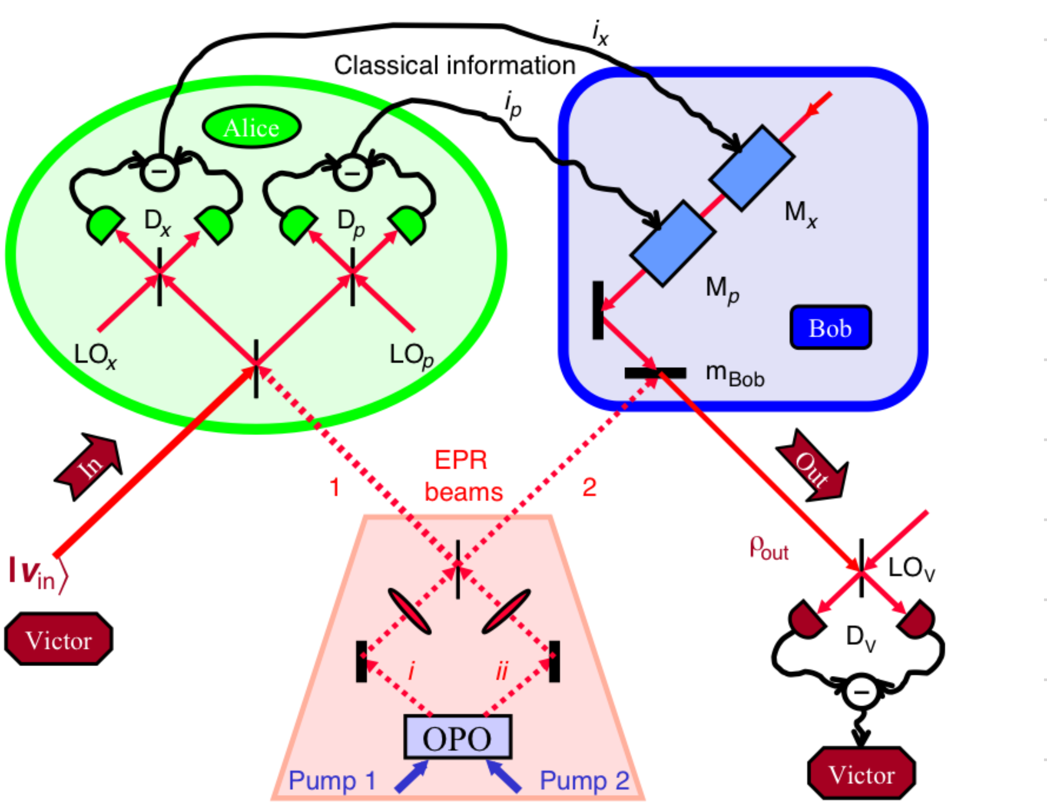
Et voila! the (unknown quantum state given to Alice by Charles) is teleported to Bob



The seminal experiment was first done in the group of Anton Zeilinger (Bouwmeester et al., Nature 390 575 (1997).). However, a full Bell state measurement was not possible, only a distinction between $|\psi_{\mp}\rangle$ and $|\phi_{\pm}\rangle$ so not all states could be teleported, and we must post-select on getting the subset of results.

The first "unconditional teleportation" we performed in the group of Jeff Kimble base on continuous variable teleportation

Fig. 1. Schematic of the experimental apparatus for teleportation of an unknown quantum state $|v_{in}\rangle$ from Alice's sending station to Bob's receiving terminal by way of the classical information (i_x, i_p) sent from Alice to Bob and the shared entanglement of the EPR beams (1, 2).



From Furusawa et al., Science 282 706 (1998).

The original protocol was presented in Braunstein & Kimble, PRL 80 869, (1998). Alice and Bob share an entangled two-mode squeezed state. This can be created in a NOPO or by mixed to single mode squeezed vacuua on a beam splitter. Alice's mode is then entangled with the mode to be teleported on another beam splitter. Position and momentum quadratures are measured (the equivalent of the Bell-state analysis of qubits). The results (real numbers) X and P are then relayed to Bob over the classical channel and he uses this to displace his mode. The result is teleportation of the mode from Alice to Bob!

To see this formally, consider the transformation on the Wigner functions

$$\underbrace{W_C(\gamma)}_{\text{charlie's state given to Alice}} \underbrace{W_{AB}(\alpha,\beta)}_{\text{two-mode EPR state}} \Longrightarrow \underbrace{W_C\left(\frac{\gamma+\alpha}{\sqrt{2}}\right) W_{AB}\left(\frac{\gamma-\alpha}{\sqrt{2}},\beta\right)}_{\text{entangling operation that mixes Charlie's mode w/ Alice.}}$$

In real Quadrature Coordinates, after A-C beam splitter

$$W_{ABC}(X_A,P_A;X_B,P_B;X_C,P_C) = W_C\left(\frac{X_A+X_C}{\sqrt{2}},\frac{P_A+P_C}{\sqrt{2}}\right) W_{AB}\left(\frac{X_C-X_A}{\sqrt{2}},\frac{P_C-P_A}{\sqrt{2}};X_B,P_B\right)$$

$$\propto W_C\left(\frac{X_A+X_C}{\sqrt{2}},\frac{P_A+P_C}{\sqrt{2}}\right)\delta\left(\frac{X_C-X_A}{\sqrt{2}}-X_B\right)\delta\left(\frac{P_C-P_A}{\sqrt{2}}+P_B\right) \quad \left(\begin{matrix}\text{for perfect}\\ \text{squeezing}\end{matrix}\right)$$

Now we measure and find $X_A=X$, $P_B=P$. The marginal state for Bob is

$$W_B(X_B,P_B) = \int dP_A \int dX_B \; W_{ABC}(X,P_A,X_B,P,X_C,P_C)$$

$$= W_C(X_B+\sqrt{2}X,\; P_B+\sqrt{2}P)$$

Thus we teleport the displaced state from Charlie to Bob via Alice. Alice sends Bob the numbers X & P and he displaces it back to obtain $W_B(X_B,P_B) = W_C(X_B,P_B)$. In practice this is limited by a number of practical factors. The squeezing is not infinite, the measurement is not a perfect projection, and there are losses. Nonetheless, fidelities exceeding classical bounds have been achieved, including the teleportation of nonclassical states with negative Wigner functions.