Physics 581: Quantum Optics II
Problem Set #1
Due Wednesday Feb. 7, 2018

Problem 1: Ambiguity of ensemble decompositions of density operators (15 points)

A density operator can be decomposed into a statistical mixture of pure states, but the decomposition is not unique. What different ensembles yield the same density operator? In this problem we prove the following:

Schrödinger-HJW Theorem: The two density operators

$$\hat{\rho}_1 = \sum_i p_i |\psi_i\rangle\langle\psi_i| \text{ and } \hat{\rho}_2 = \sum_j q_j |\phi_j\rangle\langle\phi_j|$$

are equal if and only if the two ensembles are related by,

$$\sqrt{q_j}|\phi_j\rangle = \sum_i U_{ji}\sqrt{p_i}|\psi_i\rangle,$$

where U_{ji} are elements of a partial isometry (rows and columns of U_{ji} are orthonormal).

(a) Assume the relation between the ensembles is true. Prove that $\hat{\rho}_1 = \hat{\rho}_2$.
(b) Assume $\hat{\rho}_1 = \hat{\rho}_2 \equiv \hat{\rho}$. Show $\sqrt{q_j}|\phi_j\rangle = \sum_i U_{ji}\sqrt{p_i}|\psi_i\rangle$.
(Hint: Show first that $\sqrt{p_i}|\psi_i\rangle = \sum_\alpha M_{j\alpha}\sqrt{\lambda_\alpha}|e_\alpha\rangle$, where λ_α are the eigenvalues of $\hat{\rho}$ and $|e_\alpha\rangle$ its orthonormal eigenvectors and $M_{j\alpha}$ are elements of a partial isometry. The same thus holds for $\sqrt{q_j}|\phi_j\rangle$. The proof will follow).

Problem 2: Boson Algebra (25 points)
This problem is to give you some practice manipulating the boson algebra. A great source is the classic "Quantum Statistical Properties of Radiation", by W. H. Louisell, reprinted by "Wiley Classics Library", ISBN 0-471-52365-8.

(a) Gaussian integrals in phase-space are used all the time. Show that

$$\int \frac{d^2\beta}{\pi} e^{-A|\beta|^2} e^{\alpha\beta^* - \beta\alpha^*} = \frac{1}{A} e^{-|\alpha|^2/A}.$$

(b) Prove the completeness integral for coherent states

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1} \text{ (Hint: Expand in number states).}$$

(c) The "quadrature" operators in optics are the analogs of *X* and *P*, $\hat{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}}$. Show

$$\begin{aligned} \hat{U}^\dagger(\theta)\hat{X}\hat{U}(\theta) &= \cos\theta\,\hat{X} + \sin\theta\,\hat{P} \\ \hat{U}^\dagger(\theta)\hat{P}\hat{U}(\theta) &= \cos\theta\,\hat{P} - \sin\theta\,\hat{X} \end{aligned}, \quad \text{where} \quad \hat{U}(\theta) = e^{-i\theta\hat{a}^\dagger\hat{a}}.$$

Interpret in phase space.

(d) Prove the group property of the displacement operator

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp\{i\,\mathrm{Im}(\alpha\beta^*)\}$$

(e) Show that the displacement operators has the following matrix elements

Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Coherent states: $\langle \alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-|\alpha+\alpha_2-\alpha_1|^2/2}e^{i\,\mathrm{Im}\left(\alpha\alpha_2^*-\alpha_1\alpha^*-\alpha_1\alpha_2^*\right)}$

Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\mathsf{L}_n\left(|\alpha|^2\right)$, where L_n is the Laguerre polynomial of order *n*

Problem 3: Thermal Light (25 points)

Consider a single mode field in thermal equilibrium at temperature *T*, Boltzmann factor $\beta = 1/k_B T$. The state of the field is described by the "canonical ensemble",
$\hat{\rho} = \frac{1}{Z}e^{-\beta\hat{H}}$, $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$ is the Hamiltonian and $Z = Tr\left(e^{-\beta\hat{H}}\right)$ is the partition function.

(a) Remind yourself of the basic properties by deriving the following:

- $\langle n\rangle = \frac{1}{e^{\beta\hbar\omega}-1}$ (the Planck spectrum)
- $P_n = \frac{\langle n\rangle^n}{\left(1+\langle n\rangle\right)^{n+1}}$ (the Bose-Einstein distribution).

(b) Make a list-plot of P_n for both the thermal state and the coherent state on the same graph as a function of *n,* for each of the following: $\langle n\rangle = 0.1, 1, 10, 100$.

(c) Using the number-state representation, show that

- $\Delta n^2 = \langle n\rangle + \langle n\rangle^2$. How does this compare to a coherent state?
- $\langle\hat{a}\rangle = 0 \Rightarrow \langle\vec{E}\rangle = 0$. How does this compare to a coherent state?

(d) Show that the Glauber-Sudharshan distribution of this state, $P(\alpha)=\frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle}$,

satisfies $\int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha| = \sum_n \frac{\langle n\rangle^n}{\left(1+\langle n\rangle\right)^{n+1}}|n\rangle\langle n|$. Sketch $P(\alpha)$ in the phase plane.

Problem 4: Twin beams and two-mode squeezed states. (20 points)

Considering the Hamiltonian

$$\hat{H} = i\hbar G\left(\hat{a}_+^\dagger \hat{a}_-^\dagger e^{-i\phi} - \hat{a}_+\hat{a}_- e^{i\phi}\right),$$

where $\hat{a}_\pm$ are annihilation operators for two modes with frequencies $\omega_\pm$. We will see in class how this arises in nonlinear optics through the process of parametric down-conversion. This leads to correlated twin "signal" and "idler" beams as long as the phase matching conditions are satisfied,

$$\omega_p = \omega_s + \omega_i,\ \ \mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i.$$

Here G is the coupling constant depending on the nonlinearity, pump amplitude, and vacuum mode strength. The state produced is known as a "two-mode squeezed vacuum state", $\hat{S}_\pm(\xi)|0\rangle_+ \otimes |0\rangle_- = \exp\left[\xi\hat{a}_+\hat{a}_- - \xi^*\hat{a}_+^\dagger\hat{a}_-^\dagger\right]|0\rangle_+ \otimes |0\rangle_-$, where $\xi = re^{i\phi}$ is the complex squeezing parameter for an interaction time *t*, $r = Gt$.

(a) Show that the generalized Bogoliubov transformations is

$$\hat{S}_\pm^\dagger(\xi)\hat{a}_\pm\hat{S}_\pm(\xi) = \cosh(r)\hat{a}_\pm - e^{-i\phi}\sinh(r)\hat{a}_\mp^\dagger.$$

(b) Show that the individual modes, $\hat{a}_\pm$, show no squeezing, but that squeezing exists in the *correlation* between the modes. Hint: consider quadratures,

$$\hat{X}_\pm(\theta) \equiv \frac{\hat{a}_\pm e^{i\theta} + \hat{a}_\pm^\dagger e^{-i\theta}}{2} \text{ and then } \hat{Y}(\theta,\theta') \equiv \left(\hat{X}_+(\theta) - \hat{X}_-(\theta')\right)/\sqrt{2}.$$

For the remaining parts, take ξ real.

(c) The two-mode squeezed state is an entangled state between the signal and idler as we know from the perturbative limit of twin photons. Show that in the Fock basis

$$\hat{S}_{\pm}(r)|0\rangle_{+}\otimes|0\rangle_{-}=\left(\cosh(r)\right)^{-1}\sum_{n=0}^{\infty}\left(\tanh(r)\right)^{n}|n\rangle_{+}\otimes|n\rangle_{-}.$$

Hint: Use the "disentangling theorem" (D. R. Traux, Phys. Rev. D **31**, 1988 (1985)):

$$e^{r\left(\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger}-\hat{a}_{+}\hat{a}_{-}\right)}=e^{\Gamma\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger}}e^{-g\left(\hat{a}_{+}^{\dagger}\hat{a}_{+}+\hat{a}_{-}^{\dagger}\hat{a}_{-}+1\right)}e^{-\Gamma\hat{a}_{+}\hat{a}_{-}}.$$
$$\text{where }\Gamma=\tanh(r),\ g=\ln\left(\cosh(r)\right)$$

The photons are produced with perfect correlations between the modes. This is known as "number squeezing" in "twin beams.