

Physics 581, Quantum Optics II
Problem Set #3
Due: Wednesday March 7, 2018

Problem 1: Some more boson Algebra (20 Points)

(a) Show that the displacement operators are orthogonal according to the Hilbert-Schmidt inner product, $Tr(\hat{D}^\dagger(\alpha)\hat{D}(\beta)) = \pi\delta^{(2)}(\alpha - \beta)$.

Hint: Recall $Tr(\hat{A}) = Tr\left(\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| \hat{A}\right) = \int \frac{d^2\alpha}{\pi} \langle\alpha|\hat{A}|\alpha\rangle$

(b) We have shown that the Fourier transform of the displacement operators are

$$\hat{T}_\sigma(\alpha) \equiv \int \frac{d^2\beta}{\pi} \hat{D}_\sigma(\beta) e^{\alpha\beta^* - \alpha^*\beta} = \pi \left\{ \delta^{(2)}(\alpha - \hat{a}) \delta^{(2)}(\alpha^* - \hat{a}^\dagger) \right\}_\sigma.$$

Show that $\hat{T}_{-1}(\alpha) = |\alpha\rangle\langle\alpha|$ (Hint: Insert $\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|$ appropriately)

(c) Show that for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, the Wigner function is

$$W(X,P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi} \psi^*\left(X + \frac{Y}{2}\right) \psi\left(X - \frac{Y}{2}\right) e^{-iPY}, \text{ where } W(X,P) = \frac{1}{2} W(\alpha).$$

(d) Show that the Wigner function yields the correct marginals in X and P,

$$\int_{-\infty}^{\infty} dP W(X,P) = |\psi(X)|^2, \quad \int_{-\infty}^{\infty} dX W(X,P) = |\tilde{\psi}(P)|^2,$$

and for an arbitrary quadrature

$$\int_{-\infty}^{\infty} dP_\theta W(X,P) = |\tilde{\psi}(X_\theta)|^2$$

Problem 2: Calculation of some quasiprobability functions (25 points)

(a) Find the P , Q , and W distributions for a thermal state

$$\hat{\rho} = \frac{e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_B T}}{Z}, Z = \text{Tr}(e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_B T}) = \text{partition function}$$

and show they are *Gaussian* functions. For example, you should find

$$P(\alpha) = \frac{1}{\pi\langle n \rangle} \exp\left(-\frac{|\alpha|^2}{\langle n \rangle}\right). \text{ Show that these three distributions give the proper functions in}$$

the limit, $\langle n \rangle \rightarrow 0$, i.e. the vacuum.

(b) Find the P , Q , and W distributions squeezed state $|\psi\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$. In what sense is this state nonclassical?

(c) Find the Glauber-Sudarshan P-representation for a Fock state $|\psi\rangle = |n\rangle$. Comment.

(d) Consider a superposition state of two “macroscopically” distinguishable coherent states,

$$|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle), |\alpha_1 - \alpha_2| \gg 1, \text{ where } N = \left[2(1 + \exp\{-|\alpha_1 - \alpha_2|^2\})\right]^{-1/2} \text{ is normalization.}$$

This state is often referred to as a “Schrodinger cat”, and is very nonclassical. Calculate the Wigner function, for the case $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$, with α real, and plot it for different values of $|\alpha_1 - \alpha_2| = 2\alpha$. Comment please.

(e) Calculate the marginals of the Schrodinger-cat Wigner function in X and P and show they are what you expect.

Problem 3: An Alternative Representation of the Wigner Function. (20 points)

We have shown that Wigner function could be expressed as

$$W(\alpha) = \frac{1}{\pi} \text{Tr}(\hat{\rho}\hat{T}(\alpha)) = \frac{1}{\pi} \langle \hat{T}(\alpha) \rangle, \text{ where } \hat{T}(\alpha) = \int \frac{d^2\beta}{\pi} \hat{D}(\beta) e^{\alpha\beta^* - \beta^*\alpha}$$

(a) Show that $\hat{T}(\alpha) = \hat{D}(\alpha)\hat{T}(0)\hat{D}^\dagger(\alpha)$.

(b) Show that $\hat{T}(0) = 2(-1)^{\hat{a}^\dagger\hat{a}}$. (This is a tough problem. You may assume the answer and work backwards or try to find a direct proof).

Note: the operator $(-1)^{\hat{a}^\dagger \hat{a}} = \sum_n (-1)^n |n\rangle \langle n| = \int dX |-X\rangle \langle X|$ is the “parity operator” (+1 for even parity, -1 for odd parity). Thus we see that the Wigner function at the origin is given by the expected value of the parity.

$$W(0) = \frac{2}{\pi} \text{Tr} \left[\hat{\rho} (-1)^{\hat{a}^\dagger \hat{a}} \right] = \frac{2}{\pi} \sum_n (-1)^n \langle n | \hat{\rho} | n \rangle.$$

(c) Show that general expression

$$\hat{T}(\alpha) = 2 \hat{D}(\alpha) (-1)^{\hat{a}^\dagger \hat{a}} \hat{D}^\dagger(\alpha) = 2 \sum_n (-1)^n \hat{D}(\alpha) |n\rangle \langle n| \hat{D}^\dagger(\alpha),$$

$$\text{and thus } W(\alpha) = \frac{2}{\pi} \sum_n (-1)^n \langle n | \hat{D}^\dagger(\alpha) \hat{\rho} \hat{D}(\alpha) | n \rangle.$$

This expression provides a way to “measure” the Wigner function. One displaces the state to the point of interest, $\hat{D}^\dagger(\alpha) \hat{\rho} \hat{D}(\alpha)$, one then measures the photon statistics $p_{n\alpha} = \langle n | \hat{D}^\dagger(\alpha) \hat{\rho} \hat{D}(\alpha) | n \rangle$. Putting this in the parity sum gives $W(\alpha)$ at that point!

This is a form a quantum-state reconstruction, also know as “quantum tomography,” which we will study in a future problem.