Physics 581, Quantum Optics II

Problem Set #5 (EXTRA CREDIT)
Due: Wednesday April 11, 2018

Problem 1: The EPR state (EXTRA CREDIT: 30 points)
The Einstein-Podolsky-Rosen (EPR) paradox is based around a thought experiment of measurements on an entangled state of the motion of two particles. The EPR state is a simultaneous eigenstate of relative position and the center-of-mass momentum

$$\left(\hat{X}_A - \hat{X}_B\right)|EPR\rangle = X_{rel}|EPR\rangle \;,\; \left(\hat{P}_A + \hat{P}_B\right)|EPR\rangle = P_{com}|EPR\rangle$$

The purpose of this problem is to show how one can create an approximation to this state in quantum optics, and to study their entanglement properties.

(a) We showed that the photon pair produced in spontaneous parametric down conversion was entangled in frequency and time of emission. By selecting a narrow pinhole of phase-matched signal (*s*) and idler (*i*) directions, the state can be written

$$|\Psi\rangle = \int d\omega_s\, \tilde{A}(\omega_s)\,|\omega_s\rangle_s \otimes |\omega_p - \omega_s\rangle_i = \int dt'\, A(t-t')\,|t\rangle_s \otimes |t'\rangle_i \,.$$

Here, $\tilde{A}(\omega_s)$ is the spectrum of signal frequencies allowed through the pinhole, $|\omega\rangle_{s(i)}$ is a mode with frequency ω travelling in the *s*(*i*) direction, and $|t\rangle_{s(i)}$ is a "temporal mode" representing a photon localized near position c*t* along the beam.

Argue that in the limit, $\tilde{A}(\omega_s) \to 1/2\pi$, $A(t-t') \to \delta(t-t')$*, this is the EPR state. What plays the role of position and momentum?*

Consider now a parametric oscillator beyond the perturbative limit, where two modes (*A* and *B*) are phase matched with the pump. The resulting output state is a two-mode squeezed vacuum state

$$|0,0\rangle_r = \hat{S}_{AB}(r)|0\rangle_A|0\rangle_B = e^{r\left(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}\right)}|0\rangle_A|0\rangle_B$$

Our goal is to show that in the limit of infinite squeezing, the is the EPR state.

(b) Show that, $\hat{S}^\dagger_{AB}\left(\hat{P}_A \pm \hat{P}_B\right)\hat{S}_{AB} = \left(\hat{P}_A \pm \hat{P}_B\right)e^{\mp r}$, and thus this operation squeezes the "relative position" and "center-of-mass momentum" quadratures.

(c) Show that $\hat{S}^\dagger_{AB}\hat{X}_A\hat{S}_{AB} = \cosh r\hat{X}_A + \sinh r\hat{X}_B$, $\hat{S}^\dagger_{AB}\hat{X}_B\hat{S}_{AB} = \cosh r\hat{X}_B + \sinh r\hat{X}_A$. This is a Heisenberg statement.

(d) From this argue that, up to normalization (which is tricky position)

$$\hat{S}_{AB}(r)\left|X_A\right\rangle_A\left|X_B\right\rangle_B = \left|\cosh r X_A + \sinh r X_B\right\rangle_A \left|\cosh r X_B + \sinh r X_A\right\rangle_B$$

(e) Show that the (normalized) position space wave function for the two modes is

$$\Psi_r(X_A, X_B) = \langle X_A|\langle X_B\|0,0\rangle_r = \frac{1}{\sqrt{\pi}} e^{-\frac{(X_A - X_B)^2}{4e^{-2r}}} e^{-\frac{(X_A + X_B)^2}{4e^{+2r}}} \quad \text{(plot for } r\text{=2)}$$

and in the limit of infinite squeezing $\lim_{r\to\infty}\Psi_r(X_A, X_B) \Rightarrow \delta(X_A - X_B)$

(f) By similar arguments, show that the (normalized) momentum space wave function is

$$\tilde{\Psi}_r(P_A, P_B) = \langle P_A|\langle P_B\|0,0\rangle_r = \frac{1}{\sqrt{\pi}} e^{-\frac{(P_A + P_B)^2}{4e^{-2r}}} e^{-\frac{(P_A - P_B)^2}{4e^{+2r}}} \quad \text{(plot for } r\text{=2)}$$

and in the limit of infinite squeezing $\lim_{r\to\infty}\tilde{\Psi}_r(P_A, P_B) \Rightarrow \delta(P_A + P_B)$

Thus argue that in the limit of infinite squeezing, the two-mode squeezed vacuum is the EPR state.

(g) Show that in the limit of infinite squeezing, the two-mode squeezed state can be expressed as

$$\lim_{r\to\infty}|0,0\rangle_r \Rightarrow |EPR\rangle = \int dX\,|X\rangle_A \otimes |X\rangle_B = \int dP\,|P\rangle_A \otimes |-P\rangle_B = \sum_n |n\rangle_A \otimes |n\rangle_B$$

Note: This is maximally entangled state in infinite dimensions. It is not a physical state, however, as it requires infinite energy. Nonetheless, we approximate is with large, but finite squeezing.

(h) Show that the Wigner function for the two-mode state is

$$W(X_A, P_A, X_B, P_B) = \left|\Psi_r(X_A, X_B)\right|^2 \left|\tilde{\Psi}_r(P_A, P_B)\right|^2 = \frac{1}{\pi^2} e^{-\frac{(X_A - X_B)^2 + (P_A + P_B)^2}{2e^{-2r}}} e^{-\frac{(X_A + X_B)^2 + (P_A - P_B)^2}{2e^{+2r}}}$$

(i) Extra credit: The Wigner function is positive, meaning there is a classical local probabilistic description of joint measurements of X_A, X_B, P_A, P_B. What are the implications for the EPR paradox and Bell's inequalities?

Problem 2: Gaussian States in Quantum Optics (EXTRA CREDIT: 35 points)

The set of states whose quadrature fluctuations are Gaussian distributed about a mean value is an important class in quantum optics. These states have Gaussian Wigner functions. In this problem, we explore Gaussian states, their relationship to squeezing, and the canonical algebra of phase space.

Consider a field of *n-modes*, with quadrature defined by an ordered vector:

$$\mathbf{Z} = (X_1, P_1, X_2, P_2, ..., X_n, P_n).$$

The operators associated with these quadratures satisfy a set of canonical commutators relations that can be written compactly as,

$$\left[\hat{Z}_i, \hat{Z}_j\right] = \frac{i}{2}\Sigma_{ij}, \text{ where } \Sigma = \bigoplus_{k=1}^{n}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix.}$$

We define an "inner product" in phase space as $\left(\mathbf{Z}|\mathbf{Q}\right) = Z_i \Sigma_{ij} Q_j$ (summed over repeated indices through this problem).

(a) Show that the phase space displacement operator can be written

$$\hat{D}(\mathbf{Z}) = \exp\left\{-i\left(\mathbf{Z}|\hat{\mathbf{Z}}\right)\right\}$$

A *Gaussian state* is one whose Wigner function is a Gaussian function on phase space. Recall the characteristic function of a quantum state is defined $\chi(\mathbf{Z}) = Tr\left(\hat{\rho}\hat{D}(\mathbf{Z})\right)$.
The general form of the characteristic function for a Gaussian state with is:

$$\chi(\mathbf{Z}) = \exp\left\{-\frac{1}{2}\left(\mathbf{Z}|\mathbf{C}|\mathbf{Z}\right) + i\left(\mathbf{d}|\mathbf{Z}\right)\right\}.$$

Where C_{ij} is known as the covariance matrix, and d_i is a real vector.

(b) Show that: $\left\langle \hat{Z}_i \right\rangle = d_i$, and $\frac{1}{2}\left\langle \Delta\hat{Z}_i \Delta\hat{Z}_j + \Delta\hat{Z}_j \Delta\hat{Z}_i \right\rangle = C_{ij}$, where $\Delta\hat{Z}_i \equiv \hat{Z}_i - \left\langle \hat{Z}_i \right\rangle$.
Hint: Recall how moments are found from the characteristic function.

The Gaussian state is thus determined by the mean position in phase space and the covariance of all the fluctuations.

(c) Find the Wigner function for a state with the general form of the characteristic function.

Let us restrict our attention to Gaussian states with zero mean (the mean is irrelevant to the statistics and can always be removed via a displacement operation). Consider now unitary transformations on the state. A particular class of transformations is the set that act as linear canonical transformations, i.e.

$$\hat{U}^\dagger \hat{Z}_i \hat{U} = S_{ij}\hat{Z}_j \text{, where } S_{ij} \text{ is a symplectic matrix, defined by } S^T \Sigma S = \Sigma .$$

A unitary map on the state transforms the state according to

$$\chi(\mathbf{Z}) \Rightarrow \chi'(\mathbf{Z}) = Tr\left(\hat{U}\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\right) = Tr\left(\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\hat{U}\right).$$

(d) Show that for a symplectic transformation, the characteristic function transforms as

$$\chi(\mathbf{Z}) \Rightarrow \chi(\mathbf{SZ})$$

and thus the action of the unitary is to *preserve the Gaussian statistics, by* transforming covariance matrix as $\mathbf{C} \Rightarrow \mathbf{S}^T\mathbf{CS}$.

(e) Show that the following operations preserve Gaussian statistics:

- Linear optics: $\hat{U} = \exp\left(-i\theta_{ij}\hat{a}_i^\dagger \hat{a}_j\right)$
- Squeezing: $\hat{U} = \exp\left(\zeta_{ij}^* \hat{a}_i \hat{a}_j - \zeta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger\right)$

(f) For each of these, show how the covariance matrix of the Gaussian transforms.

(g) Starting with the vacuum (a Gaussian state) we apply the squeezing operator above. Show that the symplectic transformation on the covariant matrix leads to the expected result.