

Physics 581 - Quantum Optics II

Lecture 4: Quasiprobability Functions of the Field

Motivation

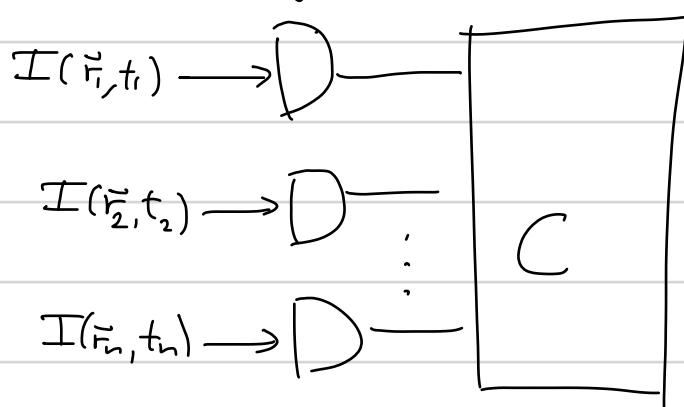
We have defined "classical light" as that whose state can be expressed as a statistical mixture of coherent states

$$\hat{P} = \int d^2\{\alpha_k\} P(\{\alpha_k\}) |\{\alpha_k\}\rangle \langle \{\alpha_k\}|$$

$P(\{\alpha_k\})$ = Glauber-Sudarshan "P-representation." Classical $\Rightarrow P(\{\alpha_k\}) \geq 0$

This definition of classicality is defined by the condition that a photon counting experiment is describable by the semiclassical theory (i.e. classical wave inducing photo-electric emission).

E.g. Coincidence counting from n-detectors



$$C = \langle \overset{\bullet}{\underset{P}{\dot{\mathcal{I}}}}(\vec{r}_1, t_1) \overset{\dot{\mathcal{I}}}{\underset{P}{\dot{\mathcal{I}}}}(\vec{r}_2, t_2) \dots \overset{\dot{\mathcal{I}}}{\underset{P}{\dot{\mathcal{I}}}}(\vec{r}_n, t_n) \rangle$$

normal order

$$= \sum \langle \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \dots \hat{a}_{k_n}^\dagger \hat{a}_1 \hat{a}_2 \dots \hat{a}_n \rangle$$

$$\rho_{\text{classical}} \rightarrow C = \overline{|E_1|^2 |E_2|^2 \dots |E_n|^2}$$

classical statistical average

$$= \sum \int d^2\{\alpha_k\} P(\{\alpha_k\}) |\alpha_1|^2 |\alpha_2|^2 \dots |\alpha_n|^2$$

Any state of the field arising from a classical (maybe noisy) current source is described by a statistical mixture of coherent states, so in this sense the light is classical.

We can also, however, consider other detection methods, and ask whether there is a classical, statistical description of the process.

Consider, e.g., homodyne detection:

$$I_{out} \propto \hat{X}_\phi = \frac{e^{-i\phi}\hat{a} + e^{i\phi}\hat{a}^\dagger}{\sqrt{2}}$$

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \right)^2 \right\rangle = \frac{1}{2} \underbrace{\left\langle \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \right\rangle}_{\text{symmetric order}}$$

A squeezed state is non-classical in the sense that it cannot be represented as a statistical mixture of coherent states (and thus does not arise as the radiation of a classical noise source), but note

$$\langle \hat{X}^2 \rangle = \int dx \frac{e^{-\frac{x^2}{2\Delta x^2}}}{\sqrt{2\pi\Delta x^2}} X^2 = \text{classical statistical}$$

average, even when $\Delta x^2 < \frac{1}{2}$ (vacuum limit). So in some sense, the squeezed vacuum can be understood in a classical statistical noise theory.

That is, there exists another way of representing the state in terms of coherent states, such that

$$\langle \hat{X}^2 \rangle = \left\langle \left(\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \right)^2 \right\rangle = \int d\omega W(\omega) \left(\frac{\alpha + \alpha^*}{\sqrt{2}} \right)^2$$

$W(\omega)$ ≡ "Wigner function"

Quantum Mechanics in Phase Space

The fact that there are many different ways to represent a state in terms coherent states follows from the fact that they are "phase space eigenstates."

$$(\hat{X} + i\hat{P}) |\alpha\rangle = (X + iP) |\alpha\rangle \quad X = \sqrt{2} \operatorname{Re}(\alpha) \\ P = \sqrt{2} \operatorname{Im}(\alpha)$$

$|\alpha\rangle$ is not an eigenstate of a Hermitian operator, and thus, they are not orthogonal, $|\langle \alpha | \alpha' \rangle|^2 = e^{-|\alpha - \alpha'|^2} \Rightarrow \{|\alpha\rangle\}$ is an over-complete basis: $\int d\alpha |\alpha\rangle \langle \alpha| = \mathbb{1}$, but smaller sets will do.

The question of how to represent quantum states in phase space has a long history. From a fundamental perspective, it allows us to compare classical and quantum behavior. Classically, we can formulate Hamiltonian dynamics via Liouville theory.

Classical State: Probability distribution in phase space $W(\vec{q}, \vec{p}, t)$

Liouville equation for dynamics:

$$\frac{d\vec{p}}{dt} = - \underset{\text{Poisson bracket}}{\{H(\vec{q}, \vec{p}), W(\vec{q}, \vec{p}, t)\}} = \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial W}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial W}{\partial q_i} \right)$$

$$\overline{f(\vec{q}, \vec{p})} \Big|_t = \int d\vec{q} d\vec{p} W(\vec{q}, \vec{p}, t) f(\vec{q}, \vec{p})$$

expected value of some observable function of the canonical coordinates.

In quantum mechanics we cannot generally represent the state as a classical probability on phase space, because \hat{X} and \hat{P} don't commute. However, use the basis of phase space eigenstates to define quasi-probability distributions such that

$$\langle f(\hat{X}, \hat{P}) \rangle_0 = \int dX dP \underbrace{W_0(X, P)}_{\text{Quasi-probability distribution}} f_0(X, P)$$

Quantum observable in phase space with ordering σ

Quasi-probability distribution for ordering σ

The issue of operating ordering arises because \hat{X} and \hat{P} don't commute

$$\begin{aligned} \text{Consider } f(\hat{X}, \hat{P}) &= (\hat{X} + \hat{P})^2 = \hat{X}^2 + \hat{P}^2 + \hat{X}\hat{P} + \hat{P}\hat{X} : \text{symmetric} \\ &= \hat{X}^2 + \hat{P}^2 + 2\hat{X}\hat{P} + i : X \text{ then } P \\ &= \hat{X}^2 + \hat{P}^2 + 2\hat{P}\hat{X} - i : P \text{ then } X \end{aligned}$$

Substituting $\hat{X} \Rightarrow X$, $\hat{P} \Rightarrow P$ gives different functions $f_0(X, P)$.

Often it is convenient to work with complex amplitudes α, α^* (quantum analogues: \hat{a}, \hat{a}^\dagger)

$$\text{Seek } W_0(\alpha) \text{ such that } \langle f(\hat{a}, \hat{a}^\dagger) \rangle_0 = \int d^2\alpha \underbrace{W_0(\alpha)}_{\text{Sign to be explained}} f_0(\alpha, \alpha^*)$$

The function $f_0(\alpha, \alpha^*)$ is defined from power series expansion

$$\text{Normal order: } f(\hat{a}, \hat{a}^\dagger) = \sum C_{nm}^{+1} (\hat{a}^\dagger)^n (\hat{a})^m$$

$$\text{Anti-normal order: } = \sum \bar{C}_{nm}^{-1} (\hat{a})^m (\hat{a}^\dagger)^n$$

$$\text{Symmetric Order} = \sum C_{nm}^0 \{(\hat{a}^\dagger)^n (\hat{a})^m\}_{\text{sym}}$$

$$\{(\hat{a}^\dagger)^n (\hat{a})^m\}_{\text{sym}} = \sum \text{Permutations of } (\hat{a}^\dagger)^n \hat{a}^m$$

$$\text{For example: } \{\hat{a}^\dagger \hat{a}^2\}_{\text{sym}} = \frac{1}{3} (\hat{a}^\dagger \hat{a}^2 + \hat{a} \hat{a}^\dagger \hat{a} + \hat{a}^2 \hat{a}^\dagger)$$

$$\text{Then } f_0(\alpha, \alpha^*) = \sum_{n,m} c_{nm}^{\sigma} (\alpha^*)^n \alpha^m$$

$$\text{Example: } f(\hat{a}^+, \hat{a}) = \hat{n}^2 = \hat{a}^+ \hat{a} \hat{a}^+ \hat{a}$$

$$= \underbrace{(\hat{a}^+)^2 (\hat{a})^2}_{C_{22}^{++} = 1} + \underbrace{\hat{a}^+ \hat{a}}_{C_{11}^{+-} = 1} = \underbrace{(\hat{a})^2 (\hat{a}^+)^2 - 3\hat{a} \hat{a}^+}_{C_{22}^{-+} = 1} + \underbrace{1}_{C_{00}^{-+} = 1}$$

$$\Rightarrow f_+(\alpha, \alpha^*) = |\alpha|^4 + |\alpha|^2, \quad f_-(\alpha, \alpha^*) = |\alpha|^4 - 3|\alpha|^2 + 1$$

The quasiprobability distribution is the representation of the density operator with respect to a given operator ordering:

$$\hat{P} = \sum c_{nm}^S \left\{ \hat{a}^{+n} \hat{a}^m \right\}_{\text{sym}} = \sum c_{nm}^N \hat{a}^{+n} \hat{a}^m = \sum c_{nm}^A \hat{a}^m \hat{a}^{+n}$$

$$W(\alpha) \equiv \sum c_{nm}^S \alpha^{*n} \alpha^m \equiv \text{Wigner Function}$$

$$Q(\alpha) \equiv \sum c_{nm}^N \alpha^{*n} \alpha^m \equiv \text{Husimi Function}$$

$$P(\alpha) \equiv \sum c_{nm}^A \alpha^{*n} \alpha^m \equiv \text{Glauber-Sudarshan P-function}$$

Weyl Group and Operator Ordering

Expressing observables as an order power series is tedious! We can achieve the operator ordering more compactly using the properties of the displacement operator (Cahill & Glauber, Phys Rev 177, 1857 (1969))

$$D(\beta) = e^{\beta \hat{a}^+ - \beta^* \hat{a}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\beta \hat{a}^+ - \beta^* \hat{a})^n$$

$$= 1 + (\beta \hat{a}^+ - \beta^* \hat{a}) + \frac{1}{2} (\beta^2 \hat{a}^{+2} - |\beta|^2 (\hat{a} \hat{a}^+ + \hat{a}^+ \hat{a}) + \beta^* \hat{a}^2) + \dots$$

$$\Rightarrow D(\beta) = \sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n! m!} \left\{ (\hat{a}^+)^n (\hat{a})^m \right\}_{\text{sym}}$$

Symmetrically ordered power series in \hat{a} and \hat{a}^+

We can use Baker-Campbell-Hausdorff $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ when c -number to express $\hat{D}(\beta)$ in normal and antinormal order:

$$\hat{D}(\beta) = e^{-|\beta|^2/2} e^{\beta \hat{a}^\dagger} e^{-\beta^* \hat{a}} = e^{-|\beta|^2/2} : \hat{D}(\beta, \beta^*) : \quad \text{normal order}$$

$$\Rightarrow : \hat{D}(\beta) : = \sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n! m!} (\hat{a}^\dagger)^n (\hat{a})^m = e^{+|\beta|^2/2} \hat{D}(\beta)$$

normally ordered power series in \hat{a} and \hat{a}^\dagger

$$\hat{D}(\beta) = e^{+|\beta|^2/2} \bar{e}^{\beta^* \hat{a}^\dagger} \bar{e}^{\beta \hat{a}^\dagger} = e^{+|\beta|^2/2} .. \hat{D}(\beta, \beta^*) .. \quad \text{anti-normal order}$$

$$\Rightarrow .. \hat{D}(\alpha) .. = \sum_{n,m} \frac{\beta^n (-\beta^*)^m}{n! m!} \hat{a}^m (\hat{a}^\dagger)^n = e^{-|\beta|^2} \hat{D}(\beta)$$

anti-normally ordered power series in \hat{a} and \hat{a}^\dagger

This construction allows us to define a "continuum" between normally ordered and antinormally ordered power series

$$\hat{D}_\sigma(\beta) = e^{\sigma |\beta|^2/2} \hat{D}(\beta)$$

$$\hat{D}_0(\beta) = e^{\beta \hat{a}^\dagger - \beta^* \hat{a}} \quad (\text{symmetric order}), \quad \hat{D}_{+1}(\beta) = e^{\beta \hat{a}^\dagger} e^{-\beta^* \hat{a}} \quad (\text{normal order})$$

$$\hat{D}_{-1}(\beta) = \bar{e}^{\beta^* \hat{a}^\dagger} \bar{e}^{\beta \hat{a}^\dagger} \quad (\text{antinormal order})$$

Characteristic function

In order to find the ordered power series representation of the density operator, we turn to a tool from classical probability theory, the "characteristic function."

Characteristic function: Fourier transform of the probability distrib.

$$\chi(y) \equiv \int_{-\infty}^{\infty} dx e^{iyx} P(x) = \overline{e^{iyx}}$$

The characteristic function is used to calculate moments of the distribution:

$$\overline{x^n} = \int_{-\infty}^{\infty} dx x^n P(x) = i^n \frac{d^n \chi}{dy^n} \Big|_{y=0}$$

For a joint probability distribution over multiple random variables

$$P(x_1, x_2, \dots, x_N) = P(\vec{x})$$

$$\chi(\vec{y}) \equiv \int d^N x e^{-i\vec{y} \cdot \vec{x}} P(\vec{x}),$$

$$\Rightarrow \overline{x_1^{n_1} x_2^{n_2} \dots x_N^{n_N}} = \frac{\partial^{n_1}}{\partial(-iy_1)^{n_1}} \frac{\partial^{n_2}}{\partial(-iy_2)^{n_2}} \dots \frac{\partial^{n_N}}{\partial(-iy_N)^{n_N}} \chi(\vec{y}) \Big|_{\vec{y}=0}$$

Weyl Representation

We seek the quasi-probability distribution on phase space, with two random variables X, P (for a single mode). We will use the idea of the characteristic function. Note that

$$\hat{D}(\beta) = D(X, P) = \underbrace{e^{-i(X\hat{P} + P\hat{X})}}_{\text{like plane wave}} = e^{\beta\hat{q}^+ - \beta^*\hat{q}^-}$$

Let $\beta = \frac{X+iP}{\sqrt{2}}$

Define characteristic function of the state of a single mode

$$\begin{aligned} \chi(X, P) &= \langle e^{-i(X\hat{P} + P\hat{X})} \rangle = \langle \hat{D}(X, P) \rangle = \text{Tr}(\hat{\rho} \hat{D}(X, P)) \\ &= \langle \hat{D}(\beta) \rangle = \text{Tr}(\hat{\rho} \hat{D}(\beta)) = \chi(\beta) \end{aligned}$$

The quasiprobability function is then defined as the inverse Fourier transform

$$W(x, p) = \int \frac{dx' dp'}{2\pi} \chi(x', p') e^{i(x'p - p'x)} = \int \frac{dx' dp'}{2\pi} \text{Tr}(\hat{\rho} \hat{D}(x', p')) e^{i(x'p - p'x)}$$

$$W(\alpha) = \int \frac{d^2 \beta}{\pi^2} \chi(\beta) e^{\alpha \beta^* - \beta \alpha^*} = \int \frac{d^2 \beta}{\pi^2} \text{Tr}(\hat{\rho} \hat{D}(\beta)) e^{\alpha \beta^* - \beta \alpha^*}$$

normalization convention

Operator ordering: $\chi(\beta) = \langle \hat{D}(\beta) \rangle = \text{symmetrically ordered product}$

General ordering:

$$W_0(\alpha) = \int \frac{d^2 \beta}{\pi^2} \chi_0(\beta) e^{-\beta^* - \alpha^* \beta}, \quad \chi_0(\beta) = \text{Tr}(\hat{\rho} \hat{D}_0(\beta))$$

$W_0(\alpha) = W(\alpha)$: Wigner function, $W_+(\alpha) = Q(\alpha)$ = Husimi function
 $W_-(\alpha) = P(\alpha)$ = Glauber-Sudarshan P-function

Weyl Algebra

To go from the characteristic function to the quasiprobability function we need the equivalent of the Fourier transform in operator space

For states, $\{|x\rangle\}$, $\{|p\rangle\}$ are complete, orthonormal bases.

$$\int_{-\infty}^{\infty} dx |x\rangle \langle x| = \int_{-\infty}^{\infty} dp |p\rangle \langle p| = \hat{1} \quad \langle x|x'\rangle = \delta(x-x') \\ \langle p|p'\rangle = \delta(p-p')$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx |x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x)} = \int_{-\infty}^{\infty} dp |p\rangle \underbrace{\langle p|\psi\rangle}_{\tilde{\psi}(p)}$$

Plane-wave / Delta function duality:

$$\langle x|x' \rangle = \delta(x-x') = \int \frac{dP}{2\pi} e^{iP(x-x')}$$

$$S^{(2)}(\alpha) = \int \frac{d^2\beta}{\pi^2} e^{\alpha\beta^* - \alpha^*\beta} \quad \alpha = \frac{x+iP}{\sqrt{2}}, \quad \beta = \frac{x'+iP'}{\sqrt{2}}$$

For operators, $\{\hat{D}(\alpha)\}$ is a complete orthonormal set of operators

$$(\hat{D}(\beta)|\hat{D}(\alpha)) = \overline{\text{Tr}}(\hat{D}^+(\beta)\hat{D}(\alpha)) = \int \frac{d^2\gamma}{\pi^2} \underbrace{\langle \gamma | \hat{D}^+(\alpha)\hat{D}(\beta) | \gamma \rangle}_{\text{coherent state}}$$

$$\text{Inner Product} = \pi S^{(2)}(\alpha - \beta) = \pi \delta(x_\alpha - x_\beta) \delta(p_\alpha - p_\beta)$$

$$\overline{\text{Tr}}(\hat{D}(\alpha)) = \pi \delta^{(2)}(\alpha)$$

$$\text{Completeness} \quad \int \frac{d^2\beta}{\pi^2} |\hat{D}(\beta)\rangle \langle \hat{D}(\beta)| = \int \frac{d^2\beta}{\pi^2} |\hat{D}^+(\beta)\rangle \langle \hat{D}^+(\beta)| = 1$$

$$\Rightarrow |\hat{A}\rangle = \int \frac{d^2\beta}{\pi^2} |\hat{D}^+(\beta)\rangle \langle \hat{D}^+(\beta)| \hat{A}$$

$$\hat{A} = \int \frac{d^2\beta}{\pi^2} \text{Tr}(\hat{D}(\beta)\hat{A}) \hat{D}^+(\beta)$$

Weyl representation

We can extend the Weyl representation to different operator orderings. Note that

$$\begin{aligned} \text{Tr}(\hat{A} \hat{D}(\beta)) \hat{D}^+(\beta) &= \text{Tr}(\hat{A} \hat{D}(\beta) e^{\sigma|\alpha|^2/2}) e^{-\sigma|\alpha|^2/2} \hat{D}^+(\alpha) \\ &= \text{Tr}(\hat{A} \hat{D}_0(\beta)) \hat{D}_{+\sigma}^+(\beta) \end{aligned}$$

$$\Rightarrow \hat{A} = \int \frac{d^2\beta}{\pi^2} \text{Tr}(\hat{A} \hat{D}_0(\beta)) \hat{D}_{+\sigma}^+(\beta)$$

Generalized Weyl representation

Finally, we have the delta-function/plane wave duality for operators.

Lemma: (to be proven in homework)

$$\hat{T}_\sigma(\alpha) = \int \frac{d^2\beta}{\pi} \hat{D}_\sigma(\beta) e^{\alpha^\ast \beta - \alpha^\ast \beta} = \overline{\text{Tr}} \left\{ \delta^{(2)}(\alpha - \hat{a}^\dagger) \delta^{(2)}(\alpha - \hat{a}) \right\}_\sigma (\text{Dual basis})$$

$$\left\{ \delta(x - \hat{x}) \delta(p - \hat{p}) \right\}_\sigma = \int \frac{dX dp'}{\pi^2} e^{i(p' \hat{x} - X \hat{p}')} e^{i(X p - p' x)}$$

(Fourier duality) Note: $\hat{T}_{-\sigma}(\alpha) = \hat{T}_\sigma^\dagger(\alpha)$, $\text{Tr}(\hat{T}_\sigma(\alpha)) = 1$

\Rightarrow The dual basis is also orthonormal and complete

$$\int \frac{d^2\alpha}{\pi} |\hat{T}_{-\sigma}(\alpha)| |\hat{T}_{+\sigma}(\alpha)| = \hat{1} \quad (\hat{T}_\sigma(\alpha) | \hat{T}_\sigma(\beta)) = \pi \delta^{(2)}(\alpha - \beta)$$

$$\Rightarrow \hat{A} = \int \frac{d^2\alpha}{\pi} \text{Tr}(\hat{T}_{-\sigma}(\alpha) \hat{A}) \hat{T}_{+\sigma}(\alpha)$$

Case of normal order: $\hat{T}_{+1} = \pi \delta(\alpha^\ast - \hat{a}^\dagger) \delta(\alpha - \hat{a}) = |\alpha\rangle \langle \alpha|$ (See homework)

Thus, as we have seen, projectors on coherent states are not orthogonal. They are orthogonal to the dual $\hat{T}_{-1}(\alpha)$.

Operator-ordered Representation

We have defined operator-ordered power series

$$\hat{A} = \underbrace{\sum_{n,m} c_{nm}^{\sigma} \{\hat{a}^{+n} \hat{a}^m\}_\sigma}_{\equiv \hat{A}_\sigma(\hat{a}^\dagger, \hat{a})} \Rightarrow \begin{aligned} &\text{O-order representation} \\ &A_\sigma(\alpha) = \hat{A}_\sigma(\hat{a} = \alpha, \hat{a}^\dagger = \alpha^\ast) \\ &= \sum_{n,m} c_{nm}^{\sigma} (\alpha^n)(\alpha^\ast)^m \end{aligned}$$

where $\{\hat{a}^{+n} \hat{a}^m\}_\sigma$ is the operator-ordered product

Normal order: $\sigma = +1$ $\{\hat{a}^{+n} \hat{a}^m\}_{+1} = (\hat{a}^\dagger)^n (\hat{a})^m$

Antinormal order: $\sigma = -1$ $\{\hat{a}^{+n} \hat{a}^m\}_{-1} = \hat{a}^m \hat{a}^{+n}$

Symmetric order: $\sigma = 0$ $\{\hat{a}^{+n} \hat{a}^m\}_0 \rightarrow$ All permutations

$$\hat{D}_\sigma(\alpha) = e^{\frac{\sigma|\alpha|^2}{2}} e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$$

$e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \quad (\sigma=+)$
 $e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \quad (\sigma=0)$
 $e^{-\alpha^* \hat{a}} \quad (\sigma=-)$
 $e^{\alpha \hat{a}^\dagger} \quad (\sigma=+)$

$$= \sum \frac{\alpha^n (-\alpha^*)^m}{n! m!} \{ \hat{a}^{+n} \hat{a}^m \}_\sigma$$

General Weyl-Representation:

$$\hat{A} = \int \frac{d^2 \beta}{\pi} \underbrace{\text{Tr}(\hat{A} \hat{D}_\sigma(\beta))}_{\tilde{A}_\sigma(\beta)} \hat{D}_\sigma^+(\beta)$$

$$\Rightarrow \hat{A} = \sum_{n,m} C_{n,m}^\sigma \{ (\hat{a}^\dagger)^n (\hat{a})^m \}_\sigma, \quad C_{n,m}^\sigma = \int \frac{d^2 \beta}{\pi} \text{Tr}(\hat{A} \hat{D}_\sigma(\beta)) \frac{\beta^n (-\beta^*)^m}{n! m!}$$

$$\Rightarrow A_\sigma(\alpha) = \int \frac{d^2 \beta}{\pi} \tilde{A}_\sigma(\beta) \hat{D}_\sigma^+(\beta) \Big|_{\substack{\hat{a}^\dagger \rightarrow \alpha \\ \hat{a} \rightarrow \alpha^*}} = \int \frac{d^2 \beta}{\pi} \tilde{A}_\sigma(\beta) e^{\alpha \beta^* - \alpha^* \beta}$$

For the specific case of the density operator, we define

$$W_\sigma(\alpha) \equiv \frac{1}{\pi} \rho_\sigma(\alpha), \quad \tilde{\rho}_\sigma(\beta) \equiv \chi_\sigma(\beta)$$

$$\Rightarrow W_\sigma(\alpha) = \int \frac{d^2 \beta}{\pi} \chi_\sigma(\beta) e^{\alpha \beta^* - \alpha^* \beta} : \text{Quasi probability}$$

Dual basis representation:

$$\begin{aligned} \hat{A}(\alpha) &= \int \frac{d^2 \beta}{\pi} \text{Tr}(\hat{A} \hat{D}_\sigma(\beta)) e^{\alpha \beta^* - \alpha^* \beta} = \text{Tr}(\hat{A} \hat{T}_\sigma^\dagger(\alpha)) \\ &= \text{Tr}(\hat{A} \{ \delta(\alpha - \beta) \delta(\alpha^* - \beta^*) \}_\sigma) \end{aligned}$$

$$\hat{A} = \int \frac{d^2 \alpha}{\pi} \text{Tr}(\hat{A} \hat{T}_\sigma^\dagger(\alpha)) \hat{T}_\sigma(\alpha) \Rightarrow \hat{A} = \int \frac{d^2 \alpha}{\pi} A_\sigma(\alpha) \hat{T}_\sigma(\alpha)$$

$$W_\sigma(\alpha) = \frac{1}{\pi} \text{Tr}(\hat{\rho} \hat{T}_\sigma^\dagger(\alpha)) = \frac{1}{\pi} \text{Tr}(\hat{\rho} \{ \delta(\alpha^* - \hat{a}^\dagger) \delta(\alpha - \hat{a}) \}_\sigma)$$

Expectation values

Consider $\langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A}) = (\hat{\rho} | \hat{A})$ (operator inner product)

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi} \text{Tr}(\hat{\rho} \hat{T}_\sigma(\alpha)) \hat{T}_\sigma(\alpha)$$

$$\Rightarrow \hat{A} = \int d^2\alpha \frac{1}{\pi} \text{Tr}(\hat{\rho} \hat{T}_\sigma(\alpha)) \text{Tr}(\hat{A} \hat{T}_\sigma(\alpha))$$

$$\Rightarrow \boxed{\langle \hat{A} \rangle = \int d^2\alpha W_\sigma(\alpha) A_\sigma(\alpha) = \int \frac{d^2\beta}{\pi} \chi_{+\sigma}(\beta) \tilde{A}_\sigma(\beta)}$$

This is the major result: Any operator function of \hat{x}, \hat{p} has different representations as functions on phase-space $A_\sigma(x, p)$

$$\langle \{\hat{a}^{+n} \hat{a}^m\}_{\text{sym}} \rangle = \int d^2\alpha (\alpha^*)^n \alpha^m W(\alpha) \quad \text{Wigner}$$

$$\langle (\hat{a}^+)^n (\hat{a})^m \rangle = \int d^2\alpha (\alpha^*)^n (\alpha)^m P(\alpha) \quad \text{Glauber}$$

$$\langle (\hat{a})^m (\hat{a}^+)^n \rangle = \int d^2\alpha (\alpha^*)^n (\alpha^m) Q(\alpha) \quad \text{Husni}$$

These moments can also be calculated from the characteristic function associated with the given ordering or ordering

$$\langle \{\hat{a}^{+n} \hat{a}^m\}_0 \rangle = \left. \frac{\partial^n}{\partial \beta^n} \frac{\partial^m}{\partial \beta^m} \chi_\sigma \right|_{\beta=0}$$

This is often the most convenient form

Properties of the quasi probability functions

Real: $W_\sigma^*(\omega) = \text{Tr}(\hat{\rho} \hat{T}_\sigma^\dagger(\omega))^* = \text{Tr}(\hat{\rho}^\dagger \hat{T}_\sigma^+(\omega)) = W_\sigma(\omega)$

Normalization: $\hat{\rho} = \int d\omega W_\sigma(\omega) \hat{T}_\sigma(\omega) = \int_{-\pi}^{\pi} \frac{d\beta}{\pi} \chi_\sigma(\beta) \hat{D}_\sigma^+(\beta)$

$$\text{Tr}(\hat{\rho}) = \int d\omega W_\sigma(\omega) \underbrace{\text{Tr}(\hat{T}_\sigma(\omega))}_1 = \int_{-\pi}^{\pi} \frac{d\beta}{\pi} \chi_\sigma(\beta) \text{Tr}(\hat{D}_\sigma^+(\beta)) \pi \delta^{(2)}(\beta)$$

$$\Rightarrow \boxed{\text{Tr}(\hat{\rho}) = \int d\omega W_\sigma(\omega) = \chi_\sigma(0)}$$

In general, however $W_\sigma(\omega) \not\geq 0 \Rightarrow$ Not probability distribution

Existence: So far we have not actually proven that and of the operator ordered power series actually converge. To prove this we turn to the Fourier analysis.

The class of function for which the Fourier transform exists are the square normalizable functions

$$\|f\|^2 = \int_{-\pi}^{\pi} |f(\omega)|^2 : \text{Finite}$$

If $\|f\|$ is finite $\tilde{f}(\beta) = \int_{-\pi}^{\pi} f(\omega) e^{i\beta^* - \omega^* \beta}$ exists, and

$$\|f\|^2 = \int_{-\pi}^{\pi} |\tilde{f}(\beta)|^2$$

Bound operators: $\|\hat{A}\|^2 = (\hat{A}|\hat{A}) = \text{Tr}(\hat{A}^\dagger \hat{A}) = \int_{-\pi}^{\pi} \hat{A}_\sigma^*(\omega) \hat{A}_\sigma(\omega)$

$\|\hat{A}\|^2 = \int_{-\pi}^{\pi} |\hat{A}_\sigma(\omega)|^2 = \int_{-\pi}^{\pi} |\tilde{A}(\beta)|^2$: Bounded if finite

Wigner function (W representation)

\hat{P} is a bounded operator $\Rightarrow \text{Tr}(\hat{P}^2) = \int d\beta |\chi_0(\beta)|^2 \leq 1$

\Rightarrow The characteristic function of Wigner function is in $L_2(\mathbb{R}^2)$

\Rightarrow $W(\alpha)$ always exists for a physical state

More generally the symmetrically ordered Weyl symbol $\hat{A}_0(\alpha)$ always exists

Husimi distribution (Q -representation)

$$Q(\alpha) = W_{+1}(\alpha) = \int \frac{d\beta}{\pi} \chi_{+1}(\beta) e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{+1}(\beta) = \text{Tr}(\hat{P} \hat{D}_{-1}(\beta)) = e^{-|\beta|^2/2} \chi_0(\beta)$$

\Rightarrow The characteristic function of Q is always square normalizable

\Rightarrow $Q(\alpha)$ always exists for a physical state

More generally the normally ordered Weyl symbol $\hat{A}_+(\alpha)$ always exists

Glauber distribution (P -representation)

$$P(\alpha) = W_{-1}(\alpha) = \int \frac{d\beta}{\pi} \chi_{-1}(\beta) e^{\alpha\beta^* - \alpha^*\beta}$$

$$\chi_{-1}(\beta) = \text{Tr}(\hat{P} \hat{D}_{+1}(\beta)) = e^{+|\beta|^2/2} \chi_0(\beta) : \text{Generally } \underline{\text{unbounded}}$$

$\Rightarrow P(\alpha)$ Only exists if $\chi_0(\beta)$ falls off at least as fast as fast as $e^{-|\beta|^2/2}$.

More generally, the antisymmetrically ordered Weyl symbol $\hat{A}_{-1}(\alpha)$ doesn't exist as a "tempered" function.

Properties of Q, P, W

Husimi Q-representation

$$Q(\alpha) = \frac{1}{\pi} \text{Tr}(\hat{\rho} \hat{T}_{-1}^{\dagger}(\alpha)) = \frac{1}{\pi} \text{Tr}(\hat{\rho} |\alpha\rangle\langle\alpha|)$$

$\Rightarrow Q(\alpha) = \frac{\langle \alpha | \hat{\rho} | \alpha \rangle}{\pi} \geq 0 \Rightarrow$ the Husimi distribution is true probability distribution

Note for a pure state $Q(\alpha) = \frac{1}{\pi} |\alpha| |\psi|^2$. This is the most natural way to define a probability distribution on phase space, as Husimi did in 1940

While $Q(\alpha)$ always exists and is always positive, it is rarely useful for making predictions. This is because

$$\langle \hat{A} \rangle = \int d^2\alpha Q(\alpha) A_{-1}(\alpha) \quad \begin{matrix} \leftarrow \text{antisymmetric} \\ \text{Weyl symbol} \end{matrix}$$

Because $A_{-1}(\alpha)$ rarely exists, this expression is not useful. That's what we expect, otherwise all of QM in phase space reduces to a classical model.

Glauber-Sudarshan P-representation

$$P(\alpha) = \frac{1}{\pi} \text{Tr}(\hat{\rho} \hat{T}_{+1}^{\dagger}(\alpha)) \Rightarrow \hat{\rho} = \int d^2\alpha P(\alpha) \hat{T}_{+1}(\alpha)$$

$$\Rightarrow \hat{\rho} = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha| : \text{Statistical mixture of coherent states}$$

$$\langle \hat{A} \rangle = \int d^2\alpha P(\alpha) \langle \alpha | \hat{A} | \alpha \rangle = \int d^2\alpha P(\alpha) A_{+1}(\alpha) \quad \begin{matrix} \leftarrow \text{always exists} \\ \text{Weyl symbol} \end{matrix}$$

$\Rightarrow P(\alpha)$ doesn't always exist, but when it does and ≥ 0 the state is essential classical

Wigner-Function W-representation

$$\langle \hat{A} \rangle = \int d\zeta W(\zeta) A_o(\zeta)$$

The Wigner Function always exists, and it always is useful for making predictions. These predictions are only equivalent to a classical statistical theory if $W(\zeta) \geq 0 \quad \forall \zeta$.

Other properties: (Proved in homework)

- Wigner's original formula: $W(X, P) = \frac{1}{2} \int_{-\infty}^{\infty} d\zeta W(\zeta) \quad$ because $\zeta = \frac{1}{\sqrt{2}}(X + iP)$

$$W(X, P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi} e^{-iPY} \langle X + \frac{Y}{2} | \hat{P} | X - \frac{Y}{2} \rangle = \int_{-\infty}^{\infty} \frac{dY}{2\pi} \psi^*(X + \frac{Y}{2}) \psi(X - \frac{Y}{2}) e^{-iPY}$$

- Marginals: (probability distribution of one variable in joint distrib)

$$P(X) = \int_{-\infty}^{\infty} dP W(X, P) = \langle X | \hat{P} | X \rangle (= |\langle X | \psi \rangle|^2 \text{ pure state})$$

$$P(P) = \int_{-\infty}^{\infty} dX W(X, P) = \langle P | \hat{P} | P \rangle (= |K_P| \psi \rangle|^2)$$

Generally, $P(X_0) = \int_{-\infty}^{\infty} dP_0 W(X, P) = \langle X_0 | \hat{P} | X_0 \rangle$ quadrature

Thus, $\langle \Delta X_0^2 \rangle = \int d\zeta \Delta X_0^2 W(\zeta)$

⇒ moments of quadratures are the moments of the Wigner function.

- Hudson-Piguet Theorem: The Wigner function of a pure state is everywhere positive (and thus quasiclassical) iff it is Gaussian $\Leftrightarrow |\psi\rangle = \hat{D}(\zeta) \hat{S}(\zeta) |0\rangle$ for some $\zeta, \zeta \Rightarrow$ squeezed/coherent state.

Relations between P, Q, W

$$W(\alpha) = \frac{2}{\pi} \int P(\beta) e^{-2|\beta - \alpha|^2} d\beta$$

$$Q(\alpha) = \frac{2}{\pi} \int W(\beta) e^{-2|\beta - \alpha|^2} d\beta = \frac{1}{\pi} \int P(\beta) e^{-|\alpha - \beta|^2} d\beta$$

$\Rightarrow W$ is a "smoothed" version of P , Q is a "smoothed" version of W . Q is the most "coarse grained" (east ripples).

Another useful formula for calculating $P(\alpha)$ and $W(\alpha)$

$$\text{Given } \hat{\rho} = \int d\alpha P(\alpha) |\alpha\rangle\langle\alpha| \Rightarrow \langle -\beta | \hat{\rho} | \beta \rangle = \int d\alpha P(\alpha) \langle -\beta | \alpha \rangle \langle \alpha | \beta \rangle$$

$$\Rightarrow \langle -\beta | \hat{\rho} | \beta \rangle = \int d\alpha P(\alpha) e^{-|\alpha|^2 - |\beta|^2} e^{\alpha \beta^* - \alpha^* \beta}$$

$$\Rightarrow \langle -\beta | \hat{\rho} | \beta \rangle \frac{e^{|\beta|^2}}{\pi} = \int \frac{d^2\alpha}{\pi} \left(P(\alpha) e^{-|\alpha|^2} \right) e^{\alpha \beta^* - \alpha^* \beta} \quad (2D \text{ Fourier transform})$$

$$\Rightarrow P(\alpha) e^{-|\alpha|^2} = \int \frac{d^2\beta}{\pi} \left(\underbrace{\langle -\beta | \hat{\rho} | \beta \rangle}_{\pi} e^{|\beta|^2} \right) e^{\alpha \beta^* - \alpha^* \beta}$$

$$\Rightarrow P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int d^2\beta \langle -\beta | \hat{\rho} | \beta \rangle e^{\alpha \beta^* - \alpha^* \beta}$$

$$W(\alpha) = \frac{2}{\pi} \int P(\beta) e^{-2|\beta - \alpha|^2} d\beta$$

$$\Rightarrow W(\alpha) = 2 \frac{e^{2|\alpha|^2}}{\pi^2} \int d^2\beta \langle -\beta | \hat{\rho} | \beta \rangle e^{2(\alpha \beta^* - \alpha^* \beta)}$$

Examples

② Coherent State $|\alpha_0\rangle$, $\rho = |\alpha_0\rangle\langle\alpha_0|$

$$Q(\alpha) = \frac{1}{\pi} |\langle\alpha_0|\alpha\rangle|^2 = \frac{1}{\pi} e^{-|\alpha-\alpha_0|^2} \Rightarrow \frac{1}{2\pi} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}} e^{-\frac{(p-p_0)^2}{2\sigma_p^2}}, \sigma_x^2 = \sigma_p^2 = 1$$

There are many routes to determining other distributions

We can read off $P(x)$ by eye

$$\hat{\rho} = \int d\alpha P(\alpha) |\alpha\rangle\langle\alpha| = |\alpha_0\rangle\langle\alpha_0| \Rightarrow P(\alpha) = \delta(\alpha - \alpha_0)$$

$$W(\alpha) = \frac{2}{\pi} \int \frac{d\beta}{\pi} P(\beta) e^{-2|\beta - \alpha|^2}$$

$$\Rightarrow W(\alpha) = \frac{2}{\pi} e^{-2|\alpha - \alpha_0|^2} \Rightarrow \frac{1}{\sqrt{\pi}\Delta x} e^{-\frac{(x-x_0)^2}{2\Delta x^2}} \frac{1}{\sqrt{\pi}\Delta p} e^{-\frac{(p-p_0)^2}{2\Delta p^2}}$$

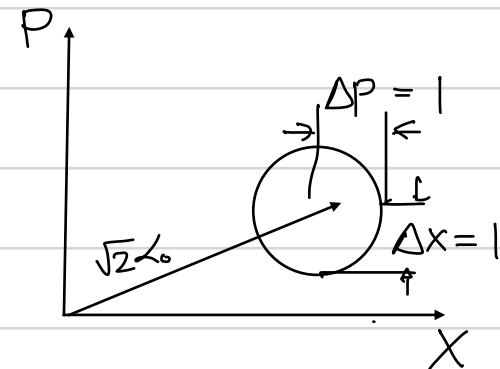
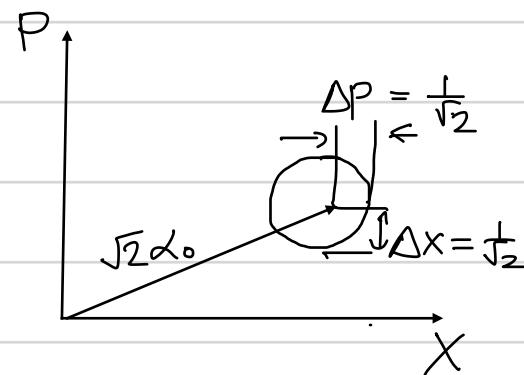
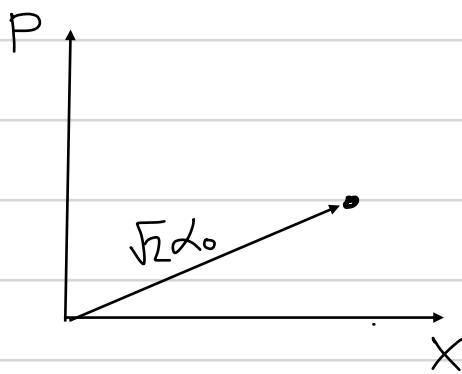
$$\Delta x = \Delta p = \frac{1}{\sqrt{2}}$$

Alternative: Characteristic function

$$\chi_0(\beta) = \text{Tr}(\hat{\rho} \hat{D}(\beta)) = \langle\alpha_0| \hat{D}(\beta)|\alpha_0\rangle = e^{-|\beta|^2/2} e^{\beta\alpha_0^* - \alpha_0^*\beta}$$

$$W_0(\alpha) = \int \frac{d\beta}{\pi} \chi_0(\beta) e^{\alpha^*\beta - \alpha^*\beta^2} \quad \text{Gaussian integral}$$

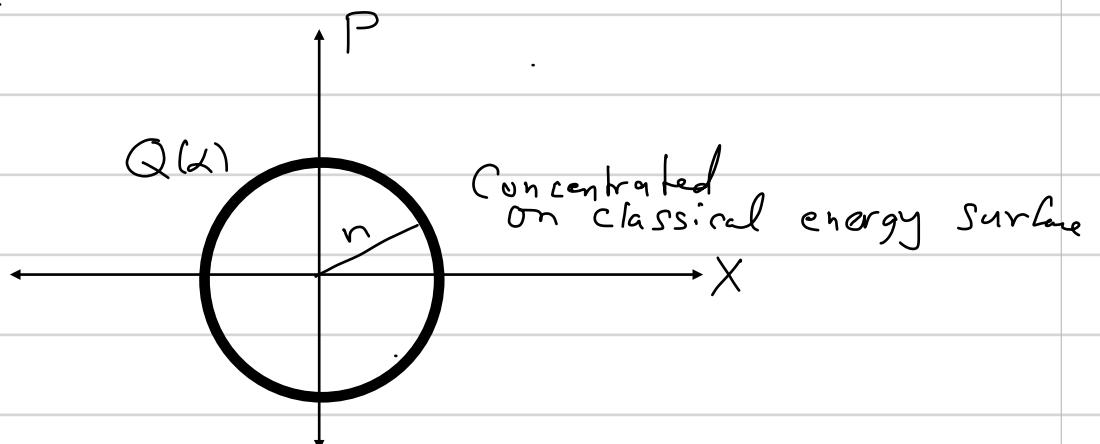
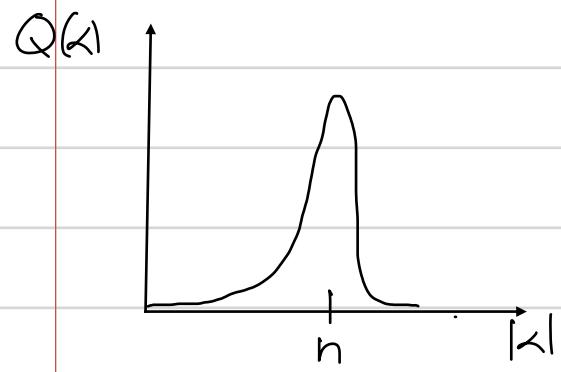
The P, W, Q are Gaussians of different widths



These "uncertainty bubbles" we have been drawing can be interpreted as the contours of the wigner function.

Fock State

$$Q(\alpha) = \frac{1}{\pi} |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{(1|\alpha|^2)^n}{\pi n!}$$



Symmetric Characteristic Function:

$$\chi_0(\beta) = \langle n | \hat{D}(\beta) | n \rangle = e^{-|\beta|^2/2} L_n^{(1)}(|\beta|^2)$$

Laguerre polynomial

$$\Rightarrow W(\alpha) = \int_{-\infty}^{\infty} \frac{d^2\beta}{\pi} \chi_0(\beta) e^{\alpha\beta^* - \beta^*\alpha} = 2(-1)^m \frac{e^{-2|\alpha|^2}}{\pi} L_n^{(4|\alpha|^2)}$$

