Physics 581, Quantum Optics II
Problem Set #3
Due: Tuesday March 1, 2022

Problem 1: Some more boson Algebra (20 Points)

(a) Show that the displacement operators are orthogonal according to the Hilbert-Schmidt inner product, $Tr\left(\hat{D}^{\dagger}(\alpha)\hat{D}(\beta)\right) = \pi\delta^{(2)}(\alpha-\beta)$.

Hint: Recall $Tr\left(\hat{A}\right) = Tr\left(\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|\hat{A}\right) = \int \frac{d^2\alpha}{\pi}\langle\alpha|\hat{A}|\alpha\rangle$

(b) Show that for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, the Wigner function is

$$W(X,P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi}\, \psi^*\left(X+\frac{Y}{2}\right)\psi\left(X-\frac{Y}{2}\right)e^{-iPY} \text{ , where } W(X,P) = \frac{1}{2}W(\alpha) \text{ .}$$

(c) Show that the Wigner function yields the correct marginals in X and P,

$$\int_{-\infty}^{\infty} dP\, W(X,P) = |\psi(X)|^2, \quad \int_{-\infty}^{\infty} dX\, W(X,P) = |\tilde{\psi}(P)|^2 \text{ , and for an arbitrary quadrature}$$

$$\int_{-\infty}^{\infty} dP_\theta\, W(X,P) = |\tilde{\psi}(X_\theta)|^2$$

Problem 2: Calculation of some quasiprobability functions (40 points)

(a) Find the *P, Q,* and *W* distributions for a thermal state

$$\hat{\rho} = \frac{e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_BT}}{Z}\text{ , } Z = Tr(e^{-\hbar\omega\hat{a}^\dagger\hat{a}/k_BT}) = \text{partition function}$$

and show they are *Gaussian* functions. For example, you should find $P(\alpha) = \frac{1}{\pi\langle n\rangle}\exp\left(-\frac{|\alpha|^2}{\langle n\rangle}\right)$. Show that these three distributions give the proper functions in the limit, $\langle n\rangle \to 0$, i.e. the vacuum.

(b) Find the *Q,* and *W* distributions for the squeezed state $|\psi\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$. Does the Glauber-Sudharshan P-resprentation exist? In what sense is this state nonclassical?

(c) Find the Q, and W distributions for a Fock state $|\psi\rangle = |n\rangle$. Find a formal expression for the Glauber-Sudharshan P-representation. This is not a well-behaved function. Comment.

(d) Consider a superposition state of two "macroscopically" distinguishable coherent states,

$|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, $|\alpha_1 - \alpha_2| >> 1$, where $N = \left[2\left(1+\exp\{-|\alpha_1 - \alpha_2|^2\}\right)\right]^{-1/2}$ is normalization.

This state is often referred to as a "Schrodinger cat", and is very nonclassical. Calculate the Wigner function, for the case $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$, with α real, and plot it for different values of $|\alpha_1 - \alpha_2| = 2\alpha$. Comment please.

(e) Calculate the marginals of the Schrödinger-cat Wigner function in X and P and show they are what you expect.

Problem 3: An Alternative Representation of the Wigner Function. (20 points)

We have shown that Wigner function could be expressed as

$$W(\alpha) = \frac{1}{\pi} Tr\left(\hat{\rho}\hat{T}(\alpha)\right) = \frac{1}{\pi}\left\langle \hat{T}(\alpha)\right\rangle, \text{ where } \hat{T}(\alpha) = \int \frac{d^2\beta}{\pi} \hat{D}(\beta) e^{\alpha\beta^* - \beta^*\alpha}$$

(a) Show that $\hat{T}(\alpha) = \hat{D}(\alpha)\hat{T}(0)\hat{D}^\dagger(\alpha)$.

(b) Show that $\hat{T}(0) = 2(-1)^{\hat{a}^\dagger\hat{a}}$. (This is a tough problem. You may assume the answer and work backwards or try to find a direct proof).

Note: the operator $(-1)^{\hat{a}^\dagger\hat{a}} = \sum_n (-1)^n |n\rangle\langle n| = \int dX |-X\rangle\langle X|$ is the "parity operator" (+1 for even parity, -1 for odd parity). Thus we see that the Wigner function at the origin is given by the expected value of the parity.

$$W(0) = \frac{2}{\pi} Tr\left[\hat{\rho}(-1)^{\hat{a}^\dagger\hat{a}}\right] = \frac{2}{\pi}\sum_n (-1)^n \langle n|\hat{\rho}|n\rangle .$$

(c) Show that general expression

$$\hat{T}(\alpha) = 2\hat{D}(\alpha)(-1)^{\hat{a}^\dagger\hat{a}}\hat{D}^\dagger(\alpha) = 2\sum_n (-1)^n \hat{D}(\alpha)|n\rangle\langle n|\hat{D}^\dagger(\alpha) ,$$

and thus $W(\alpha) = \dfrac{2}{\pi}\sum_n (-1)^n \langle n|\hat{D}^\dagger(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle$.

This expression provides a way to "measure" the Wigner function. One displaces the state to the point of interest, $\hat{D}^{\dagger}(\alpha)\hat{\rho}\hat{D}(\alpha)$, one then measures the photon statistics $p_{n\alpha} = \langle n|\hat{D}^{\dagger}(\alpha)\hat{\rho}\hat{D}(\alpha)|n\rangle$. Putting this in the parity sum gives $W(\alpha)$ at that point!

This is a form a quantum-state reconstruction, also known as "quantum tomography."