**Physics 581, Quantum Optics II**
**Problem Set #5**
**Due: Tuesday April 12, 2022**

**Problem 1: Nonclassical light generation via the Kerr effect. (15 points)**
In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\kappa}{2} : \hat{I}^2 := \frac{\hbar\kappa}{2}\hat{a}^{\dagger 2}\hat{a}^2 .$$

(a) Suppose the initial state is a coherent state $|\alpha\rangle$. *Linearize* this Hamiltonian about the mean field via the substitution $\hat{a} = \alpha + \hat{b}$, and keep terms only up to quadratic order in $\hat{b}$ and $\hat{b}^\dagger$. Show that the resulting Hamiltonian leads to squeezing. What quadrature is squeezed?

(b) Now let's go beyond the linear approximation. Show that for a long time such that $\kappa t = \pi$, the state becomes a Schrödinger cat, $\left(e^{i\pi/4}|-i\alpha\rangle + e^{-i\pi/4}|i\alpha\rangle\right)/\sqrt{2}$ (Hint: consider $i^n$ for even and odd $n$ and find the periodic pattern).

(c) (Extra Credit 10 points) Write an analytic expression for Wigner function as a function of time. Now take $\alpha$=3. Numerically make a movie of W($\alpha$,t) for $0 \leq t \leq 2\pi/\kappa$ (you will have to appropriately truncate the Fock space). Please comment on the results.

**Problem 2: Toward an optical "Schrödinger cat state." (25 points)**

Creating a "Schrödinger cat state," e.g. $\left|cat_\phi(\alpha_0)\right\rangle = N\left(|\alpha_0\rangle + e^{i\varphi}|-\alpha_0\rangle\right)$, where the normalization $N = 1/\sqrt{2\left(1+\cos\phi\, e^{-2|\alpha_0|^2}\right)}$, is a challenging task in the optical regime because we do not have sufficient nonlinearity with low loss (in the microwave regime, cavity and circuit QED has achieved this). Producing something close to such a state for applications in Quantum Information Processing has been an important goal.

Consider a squeezed single photon Fock state, $|r,1\rangle \equiv \hat{S}(r)|1\rangle$

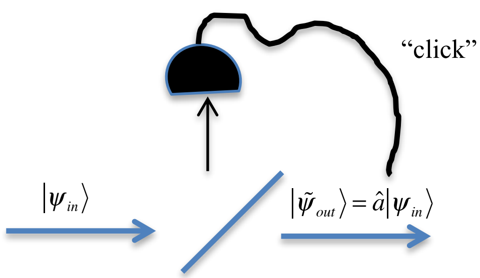
(a) Show that the Wigner function of this state is $W(\alpha) = -\frac{2}{\pi} e^{-2|b|^2} L_1(4|b|^2)$ where $b = \alpha^* \cosh r + \alpha \sinh r$ and $L_1$ is the first-order Laguerre polynomial. Plot $W$.

(b) Show that the fidelity between the "odd" cat-state and the squeezed Fock state is

$$F(r,\alpha_0,\pi) \equiv \left|\langle cat_\pi(\alpha_0)|r,1\rangle\right|^2 = \frac{2\alpha_0^2 \exp\left[\alpha_0^2(\tanh r - 1)\right]}{(\cosh r)^3\left(1-\exp[-2\alpha_0^2]\right)} \text{ (where } a_0 \text{ is real).}$$

(c) Make a surface plot of $F$ as a function of $r$ and $\alpha_0$. Under what parameters can the squeezed Fock state well approximate the cat state?

While the squeezed Fock state can approach cat state, squeezing a single photon state is not easy to achieve either. The output of a nonlinear optical processes is typically a squeezed vacuum. This is a Gaussian state, which is classical if we perform only homodyne measurements. However, if we have access to other resources, such a photon counting, we can transform this into a non-Gaussian, fully quantum resource. Consider the following experiment:




The light is incident on a highly transmitting beam splitter. Rarely one photon is reflected and detected. Conditioned on that "click," the output state has one of the photons annihilated. This state is "post selected" and the probability of producing it is rare. Nonetheless, this is a highly non-Gaussian operation.

The state produced is a "photon subtracted squeezed state." This operation is non-Unitary, so the post-measurement state is $|\psi_{out}\rangle = \hat{a}\hat{S}(r)|0\rangle / \left\|\hat{a}\hat{S}(r)|0\rangle\right\|$.

(d) Show that $|\psi_{out}\rangle = \hat{S}(r)|1\rangle$, the squeezed Fock state.

**Problem 3:** Decoherence of coherent and cat states (20 points)

Consider the damped simple harmonic oscillator, governed by the master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{\Gamma}{2}(\hat{a}^\dagger \hat{a}\rho + \rho\hat{a}^\dagger \hat{a}) + \Gamma\hat{a}\rho\hat{a}^\dagger$$

where $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$. We have studied this in the number basis. Let's study this coherent states.

(a) Suppose we prepare an initial pure coherent state, $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$. From the master equation,

$$\hat{\rho}(dt) = \left(\hat{1} - \frac{i}{\hbar}\hat{H}_{\text{eff}}dt\right)\hat{\rho}(0)\left(\hat{1} + \frac{i}{\hbar}\hat{H}_{\text{eff}}^\dagger dt\right) + \Gamma dt\ \hat{a}\hat{\rho}(0)\hat{a}^\dagger \quad \hat{H}_{\text{eff}} = \hat{H} - i\frac{\hbar\Gamma}{2}\hat{a}^\dagger\hat{a}$$

Show that $\hat{\rho}(dt) = |\alpha e^{-i\omega dt}e^{-\Gamma dt/2}\rangle\langle\alpha e^{-i\omega dt}e^{-\Gamma dt/2}|$ (hint keep terms only to order *dt*). From this argue that for finite time $\hat{\rho}(t) = |\alpha(t)\rangle\langle\alpha(t)|$, where $\alpha(t) = \alpha e^{-i\omega t}e^{-\Gamma t/2}$. I.e., the state remains a pure coherent state for all times, with a complex amplitude that follows the classical trajectory.

(b) Now consider and initial pure state which is a superposition of coherent states $|\psi(0)\rangle = \mathcal{N}\left(|\alpha\rangle + |\beta\rangle\right)$, where $\mathcal{N}^{-2} = 2(1 + \text{Re}\langle\alpha|\beta\rangle)$ is the normalization. Consider the different map generated by the master equation. Show that

$$\begin{aligned}\mathcal{A}(dt)\left[|\alpha\rangle\langle\beta|\right] &= \left(\hat{1} - \frac{i}{\hbar}\hat{H}_{\text{eff}}dt\right)|\alpha\rangle\langle\beta|\left(\hat{1} + \frac{i}{\hbar}\hat{H}_{\text{eff}}^\dagger dt\right) + \Gamma dt\ \hat{a}|\alpha\rangle\langle\beta|\hat{a}^\dagger \\ &= \langle\beta|\alpha\rangle^{\Gamma dt}\ |\alpha e^{-i\omega dt}e^{-\Gamma dt/2}\rangle\langle\beta e^{-i\omega dt}e^{-\Gamma dt/2}|\end{aligned}$$

(c) From this show that for an initial superposition of coherent states, the solution to the master equation is

$$\hat{\rho}(t) = \mathcal{N}^2\left(|\alpha(t)\rangle\langle\alpha(t)| + f(t)|\alpha(t)\rangle\langle\beta(t)| + f^*(t)|\beta(t)\rangle\langle\alpha(t)| + |\beta(t)\rangle\langle\beta(t)|\right)$$

$$\alpha(t) = \alpha e^{-i\omega t}e^{-\Gamma t/2},\ \beta(t) = \beta e^{-i\omega t}e^{-\Gamma t/2},\ f(t) = \langle\beta|\alpha\rangle^{(1-e^{-\Gamma t})}.$$

(d) For a Schrödinger cat state with $\beta = -\alpha$ and $|\alpha| >> 1$, qualitative describe the time evolution of the state.

(e) (Extra credit 10 points). Make a move of the time evolution of the cat state evolving according to master equation for three cases: $\alpha$=1,4,10. For simplicity, go into the rotating frame and by setting $\omega$=0. Comment on your results