Physics 581: Open Quantum Systems
Lecture 2: Quantum Foundations

Before we study the details of open quantum systems, let us review some of the fundamentals of closed quantum systems from will everything else is derived.

<u>Quantum Mechanics</u>: The theory which tells us how to assign probabilities to measure outcomes. That is, quantum mechanics does not tell us what "is" but only that when we observe a quantum system at some time t, we will see some measurement outcome, but we can't predict exactly what we will find. We can only assign a probability to the outcome.

The mathematical vehicle through which we assign probabilities is the "quantum state," which we assign based on the knowledge we have. If we have the maximum possible knowledge we can possibly have, consistent with quantum mechanics, then we assign a "<u>pure state</u>" $|\psi\rangle$

$|\psi\rangle \in \mathcal{H}$: Hilbert Space (complex vector space of dim d)
d can be ∞

The dimension of $\mathcal{H}$ depends on the physical degrees of freedom

Typically, when we first learn about quantum mechanics, we learn about measurements of "observables" which are specified by Hermitian operators $\hat{A}$, which have a spectral decomposition

$$\hat{A} = \sum_a a\,|a\rangle\langle a| \qquad \hat{A}\,|a\rangle = a\,|a\rangle$$

eigenvalue   eigenvector

The measurement "outcome" is labeled by the eigenvalue. For nondegenerate eigenvalues this unque (there are d-outcome), each corresponding to a corresponding eigenvector $|a\rangle$

The Born rule tells us the probability for finding a:

$$p_a = |\underbrace{\langle a|\psi\rangle}_{\text{probability amplitude}}|^2$$

And from this we have all of the statistics of the observable such as its average

$$\langle \hat{A} \rangle = \sum_a a\, p_a = \sum_a a |\langle a|\psi\rangle|^2 = \sum_a a \langle\psi|a\rangle\langle a|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle$$

We can thus write $p_a = \langle\psi|\hat{\Pi}_a|\psi\rangle$ where $\hat{\Pi}_a = |a\rangle\langle a|$: Projection operator

Note: The eigenvalue of $\hat{A}$ play no role in the measurement outcome other than labeling that outcome - it tells us which basis vector we are considering. Thus, measuring an observable, is really "measuring in a basis." The projectors form a resolution of the identity

$$\sum_{a=1}^{d} \hat{\Pi}_a = \hat{1}, \quad \text{and if } |\psi\rangle \text{ in normalized}$$

$$\sum_{a=1}^{d} p_a = 1 \rightarrow \sum_{a=1}^{d} \langle\psi|\Pi_a|\psi\rangle = \langle\psi|\psi\rangle = 1$$

A measurement in a basis is not the only measurement consistent with quantum mechanics. What is required is that for every measurement outcome we can assign a probability, which is a positive number, and we can normalize the distribution such that $\sum_a p_a = 1$

To do this, we can <u>any resolution</u> of the identity

$$\sum_{a=1}^{K} \hat{E}_a = \hat{1} \qquad \underline{\text{such that}}\ \langle\psi|\hat{E}_a|\psi\rangle \geq 0 \qquad \forall |\psi\rangle$$

The operators $\hat{E}_a$ are set to be <u>positive operators</u> $\hat{E}_a \geq 0$ and the set $\{\hat{E}_a\}$ is said to be positive operator-valued measure, or more typically a POVM for short. This is a kind of heavy handed mathematical language, but one that has stuck, so we're stuck with it

For an arbitrary POVM, we have a generalization of the Born rule

$$\boxed{p_a = \langle\psi|\hat{E}_a|\psi\rangle, \quad \text{where} \quad \sum_{a=1}^{K} \hat{E}_a = \hat{1}, \quad \hat{E}_a \geq 0}$$

Note: the number of POVM elements, K, which determines the number of possible outcomes need not be equal to d, the dimension of the Hilbert space. We will see how POVMs naturally appear in open quantum system dynamics and the measurement process.

Statistical Mixtures and the Density Operator

If we have the maximum possible information about a quantum system, we assign a pure state $|\psi\rangle$. Suppose a preparer doesn't share all of the information with us. Suppose she tells us she will flip a (possibly biased) coin, and send us the state $|\psi_1\rangle$ with probability p_1 or state $|\psi_2\rangle$ with probability p_2. If we do a measurement (say in the basis $\{|a\rangle\}$) the probability we would assign to the measurement outcome a is

$$P_a = P_{a|1}\, p_1 + P_{a|2}\, p_2 = |\langle a|\psi_1\rangle|^2 p_1 + |\langle a|\psi_2\rangle|^2 p_2$$

$$= \langle a|\psi_1\rangle\langle\psi_1|a\rangle\, p_1 + \langle a|\psi_2\rangle\langle\psi_2|a\rangle\, p_2 = \langle a|\underbrace{\left(p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|\right)}_{=\hat{\rho} \text{ Density operator}}|a\rangle$$

$$\Rightarrow \boxed{P_a = \langle a|\hat{\rho}|a\rangle} \quad \text{Born Rule}$$

That is for any measurement we might do, we find the probability of measurement outcomes from the state $\hat{\rho}$ = "density operator"

For measurement in a basis, the probability of the outcome is the diagonal matrix element associated with that basis vector.

Let us consider a pure state, expanded in a basis $\{|e_i\rangle\}$

$$|\psi\rangle = \sum_i c_i |e_i\rangle \qquad c_i = \langle e_i|\psi\rangle = \text{Probability amplitude}$$

$$\hat{\rho} = \sum_{i,j} \underbrace{c_i c_j^*}_{\rho_{ij}} |e_i\rangle\langle e_j| \qquad \rho_{ij} : \text{Elements of the density matrix}$$

$\rho_{ii} = \langle e_i|\hat{\rho}|e_i\rangle = |c_i|^2$: "Population"; probability to be found in $|e_i\rangle$

$i \neq j$ $\rho_{ij} = \langle e_i|\hat{\rho}|e_j\rangle = c_i c_j^*$: "Coherences:" the ability to see interference between $|e_i\rangle$ and $|e_j\rangle$

For example, suppose we measure in basis $\{|a\rangle\}$

$$P_a = \langle a|\hat{\rho}|a\rangle = \sum_i |c_i|^2 |\langle a|e_i\rangle|^2 + \underbrace{\sum_{i\neq j} c_i c_j^* \langle a|e_i\rangle\langle e_j|a\rangle}_{\text{illogical! Quantum interference}}$$

logic $\longrightarrow$ P_i $P_{a|i}$

This is true regarless of pure or mixed: ρ_{ii} = population, ρ_{ij} = coherence $(i \neq j)$

Note: These are basis-dependent. The size of out diagonal matrix elements depends on the choice of basis. There are other probabilities, such a purity (defined below) which are basis-independent, which quantify the degree of coherence.

Aside: The trace operation: $\mathrm{Tr}(\hat{A}) = \sum_{i=1}^{d} \langle e_i|\hat{A}|e_i\rangle$ = sum diagonal matrix elements (basis independent - sum in any basis)

Properties $\mathrm{Tr}(\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{A})$, $\mathrm{Tr}(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$, $\mathrm{Tr}(|\psi\rangle\langle\phi|\hat{A}) = \langle\phi|\hat{A}|\psi\rangle$

Thus, $P_a = \langle a|\hat{\rho}|a\rangle = \mathrm{Tr}(\hat{\rho}\,|a\rangle\langle a|) = \mathrm{Tr}(\hat{\rho}\,\hat{\Pi}_a)$ Born rule

$$\boxed{\text{Most general Born rule, POVM } \{\hat{E}_a\}, \quad P_a = \mathrm{Tr}(\hat{\rho}\,\hat{E}_a)}$$

Normalization: $\sum_a P_a = 1 = \sum_a \mathrm{Tr}(\hat{\rho}\,\hat{E}_a) = \mathrm{Tr}(\hat{\rho} \underbrace{\sum_a \hat{E}_a}_{\hat{1}})$

$\Rightarrow$ $\boxed{\mathrm{Tr}(\hat{\rho}) = 1}$ If normalized

Other Properties of Density Operator

Pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, statistical mixture $\hat{\rho} = \sum_{i=1}^{k} p_i \, |\psi_i\rangle\langle\psi_i|$

probability

$\Rightarrow$ Hermitian $\hat{\rho} = \hat{\rho}^\dagger$

$\Rightarrow$ Positive $\hat{\rho} \geq 0$ since $\langle\phi|\hat{\rho}|\phi\rangle \geq 0 \quad \forall |\phi\rangle$

Spectral decomposition: $\hat{\rho} = \sum_{\lambda=1}^{d} P_\lambda \, |\lambda\rangle\langle\lambda|$ eigenvalues $0 \leq P_\lambda \leq \lambda$

$\sum_\lambda P_\lambda = 1$

The density operator is a statistical mixture of its eigenvectors, weighted by its eigenvalues. The degree of "mixedness" or "purity" depends on the eigenvalues, and thus is a basis-independent property, unlike the "coherences"

Pure state: $P_\lambda = 1$ for some λ: $\hat{\rho} = |\lambda\rangle\langle\lambda|$

Maximally mixed state: uniform distribution $P_\lambda = \frac{1}{d} \ \forall\lambda \Rightarrow \hat{\rho} = \frac{1}{d}\sum_\lambda |\lambda\rangle\langle\lambda| = \frac{1}{d}\hat{\mathbb{1}}$

- Define the purity: $\sum_\lambda P_\lambda^2 = \mathrm{Tr}(\hat{\rho}^2)$ $\qquad \frac{1}{d} \leq \mathrm{Tr}(\hat{\rho}^2) \leq 1 \leftarrow$ Pure

maximally mixed

- The lack of knowledge about the state (mixedness) can also be quantified in an information theoretic manner. The Shannon entropy is the missing information:

$$S = -\sum_\lambda P_\lambda \log P_\lambda = -\mathrm{tr}(\hat{\rho}\log\hat{\rho}) = \text{von Neumann entropy}$$

$$\text{maximally mixed} \rightarrow \log d \leq S \leq 0 \leftarrow \text{pure state}$$

The base of the logarithm is up to us to choose. Choosing base-2, the von Neumann entropy measures the number of bits of information missing about the purity of the state.

Note: The state assignment depends on the prior information one has. In the gedanken experiment described above, the preparer (Alice), has maximum possible information. She knows whether she sent the observer (Bob) a pure state, $|\psi_1\rangle$ or $|\psi_2\rangle$. But Bob has incomplete information. His state is a statistical mixture of $|\psi_1\rangle$ and $|\psi_2\rangle$. The fact that the state assignment is different for Alice and Bob doesn't mean that the measurement outcome the occurs depends on the observer. What this means is that the degree of surprise as quantified by the probability they assign to that measurement outcome will differ.

For example, suppose Alice prepares a spin-½ particle for Bob. She tells Bob that there is a 50-50 chance she will send Bob spin-up along z or spin-down along z. Suppose if one instance, Alice sends spin-up along z. If Bob then measures along z, Alice knows exactly what will happen — he'll find spin up. But Bob doesn't have a clue. He'll find spin-up, but it was a complete surprise. If was completely random

$$\hat{\rho}_{Alice} = |\uparrow_z\rangle\langle\uparrow_z| \quad , \quad \hat{\rho}_{Bob} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z| = \frac{1}{2}\hat{\mathbb{1}}$$

Also note: The ensemble decomposition is not unique. That is, Alice can send a different set of states to Bob, with different probabilities, but from Bob's point of view these are the same state in the sense that they yield the same prediction of the probability of measurement outcomes (which is all the state can do). More formally we can define an "ensemble decomposition" $\{p_i, |\psi_i\rangle\}$ with k-elements, corresponding to the state $\hat{\rho} = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i|$. Note $\{|\psi_i\rangle\}$ need not be orthogonal, and k need not be the dimension. This is not (generally) the spectral decomposition, p_i aren't generally the eigenvalues, and $|\psi_i\rangle$ are not the eigenvectors, though the spectral decomposition is one example of an ensemble.

Two ensembles yield the same state $\sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i| = \sum_{j=1}^{l} q_j |\phi_j\rangle\langle\phi_j|$

iff $|\bar{\phi}_j\rangle = \sum_{i=1}^{k} U_{ji} |\bar{\psi}_i\rangle$ where $|\bar{\psi}_i\rangle \equiv \sqrt{p_i}\,|\psi_i\rangle$, $|\bar{\phi}_j\rangle \equiv \sqrt{q_j}\,|\phi_j\rangle$ and U_{ji} is a $l \times k$ matrix isometry. This is known as the Schrödinger–HJW theorem (we'll prove it in homework)

Dynamics:

For closed system dynamics, in the Schrödinger picture

$$\frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}\hat{H}|\psi\rangle$$

For the density operator $\hat{\rho} = \sum P_i |\psi_i\rangle\langle\psi_i|$ for some ensemble decomposition

$$\Rightarrow \frac{\partial}{\partial t}\hat{\rho} = \sum_i P_i\left[\left(\frac{\partial}{\partial t}|\psi_i\rangle\right)\left(\langle\psi_i|\right) + |\psi_i\rangle\left(\frac{\partial}{\partial t}\langle\psi_i|\right)\right] = \sum_i P_i\left(\frac{-i}{\hbar}\hat{H}|\psi_i\rangle\langle\psi_i| + \frac{i}{\hbar}|\psi_i\rangle\langle\psi_i|\hat{H}\right)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\hat{\rho} = \frac{-i}{\hbar}[\hat{H},\hat{\rho}]}$$

This is Schrödinger's equation. In this form it is sometimes known as the von Neumann eqn or the Liouville eqn - the quantum generalization.

Qubits:

One of the most important Hilbert spaces is $d=2$, which represents the state of quantum bits (qubits). The canonical example has alread been mentioned, spin-½ particles. The spin angular operator components along any direction $\hat{S}_i$ have eigenvalues $\pm\frac{\hbar}{2}$, corresponding to spin-up and spin-down along that direction. The standard basis is defined relative to the "quantization axis" z, $|\uparrow_z\rangle \equiv |0\rangle$, $|\downarrow_z\rangle \equiv |1\rangle$. We remove the $\frac{\hbar}{2}$ and define $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}} \leftarrow$ Pauli operator. The rasing and Lowering operators $\hat{\sigma}_\pm = \frac{\hat{\sigma}_x \pm i\hat{\sigma}_y}{2} = \hat{S}_\pm$ $\quad \hat{\sigma}_+ = |0\rangle\langle 1|$, $\hat{\sigma}_- = \hat{\sigma}_+^\dagger = |1\rangle\langle 0|$

Thus:

$$\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_- = |0\rangle\langle 1| + |1\rangle\langle 0| \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{\sigma}_y = \frac{\hat{\sigma}_+ - \hat{\sigma}_-}{i} = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\hat{\sigma}_z = |\uparrow_z\rangle\langle\uparrow_z| - |\downarrow_z\rangle\langle\downarrow_z| = |0\rangle\langle 0| - |1\rangle\langle 1| \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Pauli matrices = Pauli operators in standard basis

From angular momentum algebra $\quad \hat{\sigma}_i\hat{\sigma}_j = i\epsilon_{ijk}\hat{\sigma}_k + \delta_{ij}\hat{\sigma}_j$ (repeated index sum)

$\Rightarrow$

- $\hat{\sigma}_i^2 = \hat{\mathbb{1}}$
- $\hat{\sigma}_i\hat{\sigma}_j = -\hat{\sigma}_j\hat{\sigma}_i \quad (i \neq j)$
- $\mathrm{Tr}(\hat{\sigma}_i) = 0$
- $\mathrm{Tr}(\hat{\sigma}_i^2) = 2$,

Note $\hat{\sigma}_i^\dagger = \hat{\sigma}_i$ (Hermitian) and $\hat{\sigma}_i^\dagger\hat{\sigma}_i = \hat{\sigma}_i^2 = 1$ (Unitary)

Operator bases: The set of operators of a Hilbert space is itself a vector space. This most easily seen in the matrix representation. Consider, for example operators on a two-dimensional Hilbert. These are 2x2 matrice

$$\hat{A} \doteq \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{\text{"vec"}}{\Longrightarrow} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad \text{"vectorizing the matrix"}$$

We can thus map 2x2 matrices to 4-dim vectors by "stacking" the columns. Generally the set of $d \times d$ matrices is a d^2-dimensional vector space. The standard basis is on the operator space is $\hat{e}_{ij} \equiv |e_i\rangle\langle e_j|$.

For example $\hat{e}_{00} \doteq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{e}_{01} \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\hat{e}_{10} \doteq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\hat{e}_{11} \doteq \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$$\hat{A} \doteq \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\hat{e}_{00} + b\hat{e}_{01} + c\hat{e}_{10} + d\hat{e}_{11}$$

We will adopt the notation of "rounded kets" to dot the elements of thes new vector space $\hat{A} = |A) \in \mathcal{H}_{d^2}$

The vector space of matrices inherents the Euclidean inner product from the vectorized matrices. this is the Hilbert-Schmidt norm

$$(A|B) = \mathrm{Tr}(\hat{A}^\dagger \hat{B})$$

The starad basis is orthornomal

$$\mathrm{Tr}(\hat{e}_{ij}^\dagger \hat{e}_{lk}) = \mathrm{Tr}(|e_j\rangle\langle e_i|e_l\rangle\langle e_k|) = \langle e_k|e_j\rangle\langle e_l|e_i\rangle = \delta_{il}\delta_{jk}$$

The set of Pauli operators + idensity span the space of 2x2 matrices

Define $\{\hat{\sigma}_\alpha, \alpha = 0,1,2,3\}$ $\quad \hat{\sigma}_0 = \hat{\mathbb{1}}$, $\hat{\sigma}_1 = \hat{\sigma}_x$, $\hat{\sigma}_2 = \hat{\sigma}_y$, $\hat{\sigma}_3 = \hat{\sigma}_z$

Note $\mathrm{Tr}(\hat{\sigma}_\alpha^\dagger \hat{\sigma}_\beta) = \mathrm{Tr}(\hat{\sigma}_\alpha \hat{\sigma}_\beta) = \delta_{\alpha\beta}\,\mathrm{Tr}(\hat{\mathbb{1}}) = 2\delta_{\alpha\beta}$

$\Rightarrow$ $\{\hat{\sigma}_\alpha\}$ is orthogonal but not normalized.

For an arbitrary operator on $\mathcal{H}_2$

$$\boxed{\begin{aligned} \hat{A} &= \tfrac{1}{2}\sum A_\alpha \hat{\sigma}_\alpha \quad \text{where} \quad A_\alpha = \mathrm{Tr}(\hat{A}\hat{\sigma}_\alpha) \\ &= \tfrac{1}{2}\left(\mathrm{Tr}(\hat{A})\hat{\mathbb{1}} + \vec{A}\cdot\hat{\vec{\sigma}}\right) \quad A_i = \mathrm{Tr}(\hat{A}\hat{\sigma}_i) \end{aligned}}$$

Rotations and SU(2)

Rotation are generated by angular momentum. In quantum mechanics the operator that rotates a physical vector about an axis specified by unit vector $\vec{n}$, by an angle Θ is $\hat{U}_{\vec{n}}(\Theta) = e^{-i\Theta \vec{n}\cdot\hat{\vec{J}}/\hbar}$, where $\hat{\vec{J}}$ is the angular momentum operator. For spin-½ $\hat{\vec{J}} = \hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$

$$\Rightarrow \boxed{\hat{U}_{\vec{n}}(\Theta) = e^{-i\frac{\Theta}{2}\vec{n}\cdot\hat{\vec{\sigma}}}}$$

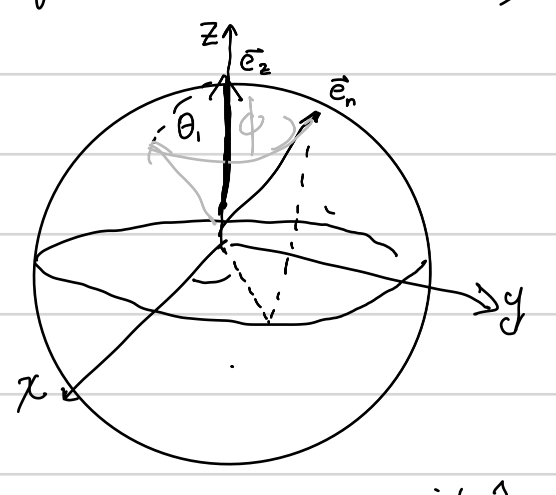
Represented as 2x2 matrices, these are unitary matrices. Moreover, since the eigenvalues of $\vec{n}\cdot\hat{\vec{\sigma}}$ are ± 1, the eigenvalues of $\hat{U}_{\vec{n}}(\Theta)$ are $e^{\pm i\Theta/2}$
$\Rightarrow \det(\hat{U}) = 1 \Rightarrow$ $\{\hat{U}(\Theta)\}$ is the group SU(2)

Using the properties of Paulis, we find

$$\boxed{\hat{U}_{\vec{n}}(\Theta) = \cos\left(\frac{\Theta}{2}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\Theta}{2}\right)\vec{n}\cdot\hat{\vec{\sigma}}}$$

A very useful formula

Consider a series of rotations in 3D that map $\vec{e}_z \to \vec{e}_n(\theta,\phi)$



$$\vec{e}_n = R_z(\phi)\,R_y(\theta)\,\vec{e}_z$$

Thus spin-up along $\vec{e}_n$: $|\uparrow_n\rangle = e^{-i\frac{\phi}{2}\hat{\sigma}_z}e^{-i\frac{\theta}{2}\hat{\sigma}_y}|\uparrow_z\rangle = e^{-i\frac{\phi}{2}\hat{\sigma}_z}\left(\cos\frac{\theta}{2}\hat{\mathbb{1}} - i\sin\frac{\theta}{2}\hat{\sigma}_y\right)|\uparrow_z\rangle$

$\Rightarrow |\uparrow_n\rangle = e^{-i\frac{\phi}{2}\hat{\sigma}_z}\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + \sin\frac{\theta}{2}|\downarrow_z\rangle\right) = e^{-i\phi/2}\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi/2}\sin\frac{\theta}{2}|\downarrow_z\rangle$

We will choose the phase convention $|\uparrow_n\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle$

similarly $|\downarrow_n\rangle = \sin\frac{\theta}{2}|\uparrow_z\rangle - e^{i\phi}\cos\frac{\theta}{2}|\downarrow_z\rangle$

Moreover, for any pure state $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle = |\alpha|\,|\uparrow_z\rangle + e^{i(\phi_\beta - \phi_\alpha)}|\beta|\,|\downarrow_z\rangle$

$|\alpha|^2 + |\beta|^2 = 1$

$\Rightarrow$ All pure states are spin-up along $\vec{n}$, where $\tan\frac{\theta}{2} = \frac{|\alpha|}{|\beta|}$, $\phi = \mathrm{Arg}(\beta) - \mathrm{Arg}(\alpha)$

Diagonalizing a 2x2 matrix

We can (and should) using the Pauli algebra to diagonalize a 2x2 matrix

We known $\hat{A} = \frac{1}{2}(Tr(\hat{A})\hat{\mathbb{1}} + \vec{A}\cdot\hat{\vec{\sigma}}) = \frac{1}{2}(Tr(\hat{A})\hat{\mathbb{1}} + |\vec{A}|\vec{e}_a\cdot\hat{\vec{\sigma}})$

where $\vec{A} = Tr(\hat{A}\hat{\vec{\sigma}})$ where $\vec{e}_a = \frac{\vec{A}}{|\vec{A}|}$

Thus immediate see: Eigenvalues of $\hat{A}$ are $\frac{1}{2}(Tr(\hat{A}) \pm |\hat{A}|)$

Eigenvector of $\hat{A}$ are $|\uparrow_a\rangle$, $|\downarrow_a\rangle$

Bloch Sphere

The set of density operators for qubits a isomorphic to the space of a unit ball in R_3. The surface of the ball is the unit sphere, Bloch sphere. We already know that the points on the unit sphere are in one-to-one correspondence with pure states $|\psi\rangle = |\uparrow_n\rangle$ $\vec{n} = (\theta, \phi)$.

More generally

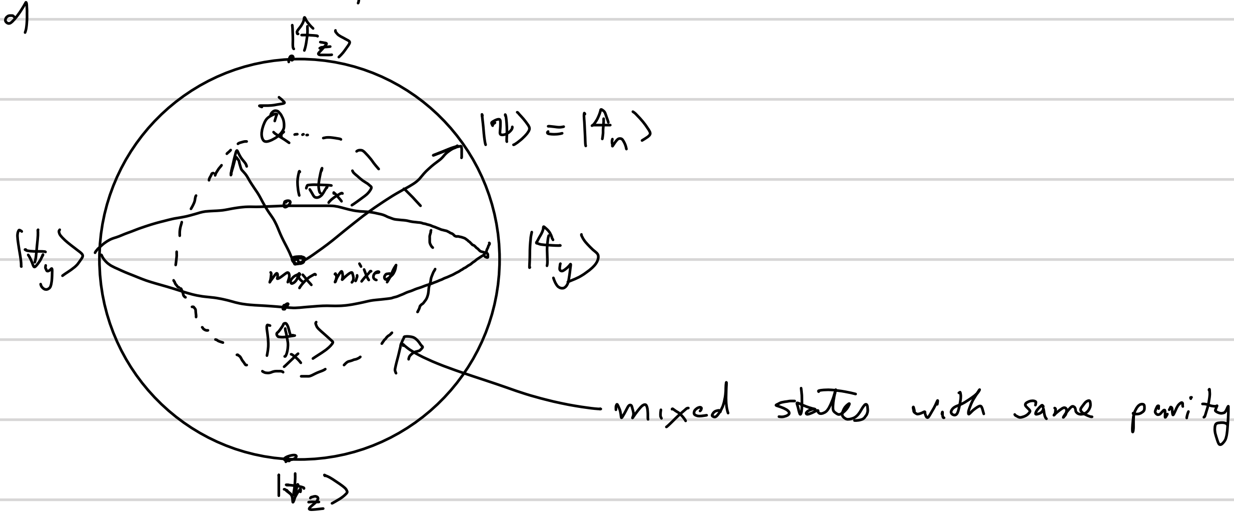
$$\hat{\rho} = \frac{1}{2}(Tr(\hat{\rho})\hat{\mathbb{1}} + Tr(\hat{\rho}\hat{\vec{\sigma}})\cdot\hat{\vec{\sigma}}) = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}\cdot\hat{\vec{\sigma}})$$

where $\vec{Q} = Tr(\hat{\rho}\hat{\vec{\sigma}}) \equiv$ Bloch vector

$$Tr(\hat{\rho}^2) = \frac{1}{2}(1 + Tr(Q_i Q_j \hat{\sigma}_i \hat{\sigma}_j)) = \frac{1}{2}(1 + |\vec{Q}|^2)$$

$\Longrightarrow$ $0 \leq |\vec{Q}| \leq 1$ ← pure state

max mixed



$|\uparrow_n\rangle\langle\uparrow_n| = \frac{\hat{\mathbb{1}} + \hat{\sigma}_n}{2}$, $|\langle\uparrow_{n'}|\uparrow_n\rangle|^2 = Tr\left[\left(\frac{\hat{\mathbb{1}} + \hat{\sigma}_{n'}}{2}\right)\left(\frac{\hat{\mathbb{1}} + \hat{\sigma}_n}{2}\right)\right] = \frac{1}{4}(Tr(\hat{\mathbb{1}}) + Tr(\hat{\sigma}_{n'}\hat{\sigma}_n))$

$$= \frac{1}{2}(1 + \vec{n}'\cdot\vec{n}) \equiv \frac{1}{2}(1 + \cos\Theta) = \cos^2\left(\frac{\Theta}{2}\right)$$

Antipodal states on Bloch-sphere $\Theta = \pi$, orthogonal.