Physics 581: Open Quantum Systems
Lecture 5: Continuous Variable Quantum Mechanics

Wave Mechanics

We have so far considered the simplest quantum system, the qubit, described by a two-dimensional Hilbert space. We now turn our attention to the opposite limit, the case of $d=\infty$ dimensional Hilbert space. This is often the first Hilbert space we encounter, discussed in the context of wave mechanics, describing the motion of nonrelativistic particle – simplest case motion in 1D.
The relevant Hilbert space is $\mathcal{L}_2(\mathbb{R})$

$$\text{square integrable functions} \quad \{\psi(x) \mid \int^{\infty} |\psi(x)|^2 dx < \infty\}.$$

The wave function $\psi(x)$ can be understood to be the position representation of a vector $|\psi\rangle$ in Hilbert space $\psi(x) = \langle x|\psi\rangle$; of dimension $d=\infty$.

Classically, we have one degree of freedom with canonical coordinates (x,p). Quantumly operators $\hat{x}$ and $\hat{p}$ are generators of translation in phase space:

$$\hat{\mathcal{J}}_{x_0} \equiv e^{-i\frac{x_0}{\hbar}\hat{p}}, \quad \text{Translation in position} \quad \hat{\mathcal{J}}^{\dagger}_{x_0}\hat{x}\,\hat{\mathcal{J}}_{x_0} = \hat{x} + x_0$$

$$\hat{m}_{p_0} = e^{+ip_0\hat{x}/\hbar}, \quad \text{translation in momentum} \quad \hat{m}^{\dagger}_{p_0}\hat{p}\,\hat{\mathcal{J}}_{p_0} = \hat{p} + p_0$$

It follows from considering infinitesimal translations that $[\hat{x},\hat{p}] = i\hbar$
$\hat{x}$ and $\hat{p}$ have eigenvectors with continuous eigenvalues

$$\hat{x}|x\rangle = x|x\rangle, \qquad \hat{p}|p\rangle = p|p\rangle$$

For this reason, particularly in quantum optics this is called "continuous variable" (CV) quantum mechanics. The position and momentum eigenvectors are not square integrable, and thus not in the Hilbert space. By they do live in the "dual" (bra) space, which allows us to write representations, and they do form a resolution of the identity

$$\int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p| = \hat{\mathbb{1}}$$, and thus POVMs

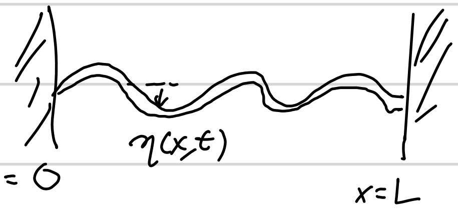
We also know $\langle x|x'\rangle = \delta(x-x')$, $\langle p|p'\rangle = \delta(p-p')$

and $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}} = \langle p|x\rangle^*$ (Plane waves)

Check $\langle x|x'\rangle = \int dp\, \langle x|p\rangle\langle p|x'\rangle = \int \frac{dx}{2\pi} e^{i(x-x')p} = \delta(x-x')$ ✓

<u>Quantum Fields</u> (Bose)

An important example where we see infinite dimensional Hilbert space is is the context of quantum fields. Consider a scalar field describing the transverse oscillations on a string, e.g. with fixed boundary conditions



$$\left(\frac{\partial^2}{\partial t} - \frac{1}{v}\frac{\partial^2}{\partial x^2}\right)\eta(x,t) = 0$$

wave equation

The field is $\eta(x,t)$ and its dynamics can be analyzed by decomposing the field into its normal modes

$$\eta(x,t) = \sum_n q_n(t)\, u_n(x) \quad \text{where} \quad u_n(x) = \sqrt{\tfrac{1}{L}}\sin(k_n x)$$

$$k_n = n\,\pi/L \qquad n = 1, 2, \ldots$$

The $q_n(t)$ are the <u>canonical coordinates</u> of the field; its modes of the degrees of freedom. To each coordinate is a canonical momentum $p_n(t)$. Following from the wave equation, the canonical coordinates all satisfy

$$\ddot{q}_n + \omega_n^2 q_n = 0 \quad \text{where} \quad \omega_n = v k_n$$

The (free) field is thus of set of simple harmonic oscillators (SHO)

The quantum field is then (in the Heisenberg picture)

$$\hat{\eta}(x,t) = \sum_n \hat{q}_n(t)\, u_n(x) \qquad [\hat{q}_n, \hat{p}_{n'}] = i\hbar\,\delta_{nn'}$$

Where the free field modes are quantum SHO. Here we are considering <u>bosons</u>, e.g. phonons for vibrations of massive motion or <u>photons</u> for vibrations of the electromagnetic field.

Simple Harmonic Oscillator

As we saw, the SHO play a very important role is field theory and the description of the dynamics of a "mode." Let us quickly review this to establish the foundation.

Classically: Motion of a massive particle in a harmonic well

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega x^2 \quad \text{(Hamiltonian)}$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \quad \Rightarrow \quad \begin{array}{l} \ddot{x} + \omega^2 x = 0 \\ \ddot{p} + \omega^2 p = 0 \end{array}$$

It is useful to make the problem dimensionless by identifying characteristic scales given the parameters. Let x_c, p_c, E_c be characteristic scales of length, momentum, and energy, so $X \equiv \frac{x}{x_c}$, $P \equiv \frac{p}{p_c}$ are dimensionless. The Hamiltonian

$$\frac{H}{E_c} = \frac{1}{2}\left(\frac{p_c^2}{mE_c}\right) P^2 + \frac{1}{2}\left(\frac{m\omega^2 x_c^2}{E_c}\right) X^2$$

We can specify x_c, p_c, E_c because we have only two parameters, m & ω.
But to make things simple, it is natural to chose $\frac{p_c^2}{mE_c} = 1$, $\frac{m\omega^2 x_c^2}{E_c} = 1$

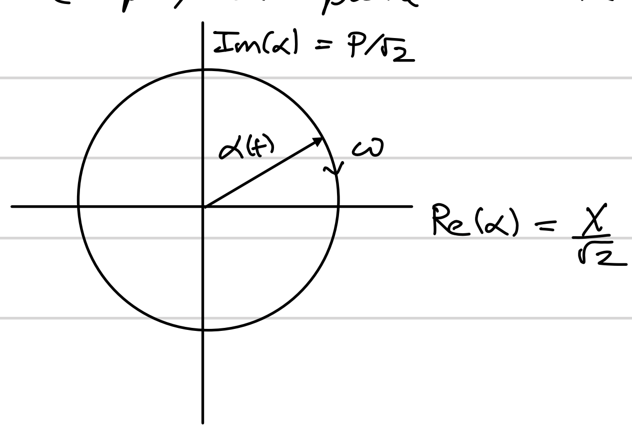
$$\Rightarrow \quad p_c = \sqrt{mE_c} \qquad x_c = \sqrt{\frac{E_c}{m\omega^2}}$$

$$\Rightarrow \quad H = E_c\left(\frac{X^2 + P^2}{2}\right), \quad \dot{X} = \omega P \quad , \quad \dot{P} = -\omega P$$

We define the complex amplitude: $\alpha \equiv \frac{X + iP}{\sqrt{2}}$, $\quad X = \frac{\alpha + \alpha^*}{\sqrt{2}}$, $\quad P = \frac{\alpha - \alpha^*}{i\sqrt{2}}$

$$\Rightarrow \dot{\alpha} = -i\omega\alpha \qquad \alpha(t) = \alpha(0)\, e^{-i\omega t} \quad , \qquad H = E_0 |\alpha|^2 \ \text{constant}$$

In dimensionless variables, the motion is phase space is equivalent to a "phasor" in the complex-α plane rotating clockwise at freq ω



Quantumly: We have a new parameter, $\hbar$, the unit of "action."
It is natural to choose $E_c = \hbar\omega$, so $p_c = \sqrt{\hbar\omega m}$, $x_c = \sqrt{\frac{\hbar}{m\omega}}$, $x_c p_c = \hbar$
Canonical coordinates $[\hat{X}, \hat{P}] = i$
Define quantized complex amplitude $\hat{a} \equiv \frac{\hat{X} + i\hat{P}}{\sqrt{2}}$, $\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}$, $\hat{P} = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}$
$[\hat{a}, \hat{a}^\dagger] = 1$

$\Rightarrow$ Quantum SHO Hamiltonian $\hat{H} = \hbar\omega\left(\frac{\hat{X}^2 + \hat{P}^2}{2}\right) = \hbar\omega\left(\frac{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}{2}\right)$
$\Rightarrow \hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{n} + \frac{1}{2})$

The eigenvectors of $\hat{H}$, are those of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$
$\hat{n}|n\rangle = n|n\rangle$ $(n = 0, 1, 2, \dots \infty)$ (A proper basis for $\mathcal{L}_2(\mathbb{R})$)

The states $|n\rangle$ have n "quanta" of excitation in the field. In field theory these quanta are the particles. Here we are Bose fields, so we can have $n > 1$ bosons occupying a given mode.

The quantized complex amplitudes are the ladder operators: annihilation/creation of quanta (particles). $\hat{a}|n\rangle = \sqrt{n}|n\rangle$, $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

It thus follows $\langle n|\hat{X}|n\rangle = \langle n|\hat{P}|n\rangle = 0$, $\langle n|\Delta X^2|n\rangle = \langle n|\Delta P^2|n\rangle = n + \frac{1}{2}$
$\Rightarrow$ For $n=0$ (vacuum of field) $\Delta X = \Delta P = \frac{1}{\sqrt{2}}$ $\Delta X \Delta P = \frac{1}{2}$

The vacuum has zero mean field, but has fluctions in the field, e.g. the electric and magnetic fields are not fixed eigenvalues of the quantum field. On average the are zero, but they fluctuate (vacuum fluctuations)
Note if $|\psi(0)\rangle = |n\rangle$ $|\psi(t)\rangle = e^{-i(n+1/2)\omega t}|n\rangle$ $\Rightarrow \langle\hat{X}\rangle = \langle\hat{P}\rangle = 0$ for all time. The energy eigenstates do not correspond to classical SH motion.

Coherent State:

We seek states such the expectation values are the classical variables, with minimum quantum uncertainty. These are the "quasiclassical" states.
Since we obtained the quantum SHO by $\alpha \to \hat{a}$, it is natural to look for eigenstates of $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \qquad |\alpha\rangle \equiv \text{coherent state}$$

then $\langle\alpha|\hat{X}|\alpha\rangle = \frac{\mathrm{Re}(\alpha)}{\sqrt{2}}$, $\langle\alpha|\hat{P}|\alpha\rangle = \frac{\mathrm{Im}(\alpha)}{\sqrt{2}}$ $\qquad \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$

Just as we have classically

Note $\langle\alpha|\hat{X}^2|\alpha\rangle = \frac{1}{2}(\langle\alpha|\hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1|\alpha\rangle = \left(\frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2 + \frac{1}{2}$

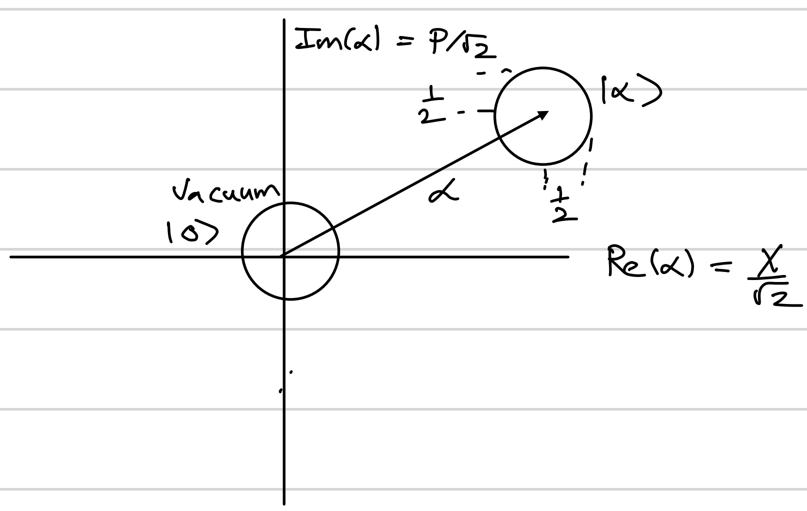
$\langle\alpha|\hat{P}^2|\alpha\rangle = \left(\frac{\alpha-\alpha^*}{\sqrt{2}\,i}\right)^2 + \frac{1}{2}$

$\Delta X = \Delta P = \frac{1}{\sqrt{2}}$ For coherent state (min uncertainty product)

$\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2 \qquad \langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle = \langle\alpha|\hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}^\dagger a|\alpha\rangle = |\alpha|^4 + |\alpha|^2$

Note, the vacuum $|0\rangle$ is also a coherent state $\hat{a}|0\rangle = 0|0\rangle = 0$ (null vector)
This is the only state which is both a Fock state and a coherent state.

We often like to draw a picture of a coherent state in phase space.
We will make this more form soon in terms of phase-space representations.



The coherent state is a wave packet centered at the phase $\alpha = \frac{\langle x\rangle + i\langle p\rangle}{\sqrt{2}}$ with fluctuations about the mean equivalent to vacuum fluctuations.

We have not yet shown the $|\alpha\rangle$ exists! $\hat{a}$ is not a normal operator, and so we cannot assume it has eigenvectors. But it does. To easily show the consider the following. As seen in the previous figure, the coherent state $|\alpha\rangle$ appears as a "displacement in phase space" of the vaccuum.

<u>Phase-space displacement</u>: Define $\hat{D}(X,P) = e^{-iX\hat{P}+i P\hat{X}}$

Note this a symmetric (in $\hat{X}$ & $\hat{P}$) version of the momentum & position translation defined on the first page. In complex amplitudes,

$$\hat{D}(X,P) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \qquad \left(\alpha = \frac{X+iP}{\sqrt{2}}\right)$$

Using Baker-Campbell-Hausdorff $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a}+\alpha$

Thus $\boxed{|\alpha\rangle = \hat{D}(\alpha)|0\rangle}$

Check: $\hat{a}|\alpha\rangle = \hat{a}\hat{D}(\alpha)|0\rangle = \hat{D}(\alpha)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)|0\rangle = \hat{D}(\alpha)(\hat{a}+\alpha)|0\rangle = \alpha|\alpha\rangle$ ✓

This form of the coherent state is very useful. From BCH

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} \quad \text{if} \quad [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$$

$$\Rightarrow \hat{D}(\alpha) = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} \quad \Rightarrow \quad |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}\underbrace{e^{-\alpha^*\hat{a}}|0\rangle}_{=|0\rangle} \quad \text{since } \hat{a}|0\rangle = 0$$

$$\Rightarrow |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_n \alpha^n \hat{a}^{\dagger n}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{\alpha_n}{\sqrt{n!}}|n\rangle$$

$$\Rightarrow \boxed{|\alpha\rangle = \sum_n c_n|n\rangle, \quad c_n = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}}$$

$$P_n = |\langle n|\alpha\rangle|^2 = |c_n|^2 = e^{-|\alpha|^2}\frac{(|\alpha|^2)^n}{n!} = e^{-\langle\hat{n}\rangle}\frac{\langle\hat{n}\rangle^n}{n!} \quad \text{Poisson distribution}$$

Also note if $|\psi(0)\rangle = |\alpha\rangle$, $|\psi(t)\rangle = e^{-i\omega t(\hat{n}+\frac{1}{2})}|\alpha\rangle = \underbrace{\cancel{e^{-\frac{i\omega t}{2}}}}_{\text{neglect}}\sum_n e^{-\frac{1}{2}|\alpha|^2}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}|n\rangle$

$\Rightarrow |\psi(t)\rangle = |\alpha e^{-i\omega t}\rangle$, coherent state that follows the classical trajectory.

More Boson Algebra

The displacement operators satisfy the composition law

$$\hat{D}(\alpha)\,\hat{D}(\beta) = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}\,\hat{D}(\alpha+\beta) = e^{i\,\mathrm{Im}(\alpha\beta^*)}\,\hat{D}(\alpha+\beta)$$

- $\{e^{i\phi}\hat{D}(\alpha)\}$ = Weyl-Heisenberg group generated by $\hat{x}, \hat{p}, \hat{1}$
- $\langle\alpha|\beta\rangle = e^{-|\alpha-\beta|^2/2}\,e^{-i\,\mathrm{Im}(\alpha^*\beta)}$, $\quad |\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$

$\Rightarrow$ The coherent states are not orthogonal, but the overlap is exponentially small when $|\alpha-\beta| \gg 1$ (separation large compared to vacuum fluctuations)

- The coherent states form an "overcomplete" basis for Hilbert space $\mathcal{L}_2(\mathbb{R})$

$$\int \frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = \hat{1}$$

- The trace of an operator can be expressed as $\mathrm{Tr}(\hat{A}) = \int \frac{d^2\alpha}{\pi}\langle\alpha|\hat{A}|\alpha\rangle$

Functions on complex plane

When dealing with CV quantum mechanics, we consider functions $f(x,p)$ over phase space, or equivalently, the complex-α plane, $f_c(\alpha,\alpha^*)$. In its complex form, we must take α and α^* to be independent variables, so when taking partial derivatives

$$\frac{\partial}{\partial x} f(x,p) = \frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha} f_c(\alpha,\alpha^*) + \frac{\partial\alpha^*}{\partial x}\frac{\partial}{\partial\alpha^*} f_c(\alpha,\alpha^*) = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*}\right) f_c(\alpha,\alpha^*)$$

$$\frac{\partial}{\partial p} f(x,p) = \frac{\partial\alpha}{\partial p}\frac{\partial}{\partial\alpha} f_c(\alpha,\alpha^*) + \frac{\partial\alpha^*}{\partial p}\frac{\partial}{\partial\alpha^*} f_c(\alpha,\alpha^*) = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\alpha^*}\right) f_c(\alpha,\alpha^*)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}\right) f(x,p) = 2\frac{\partial^2}{\partial\alpha\,\partial\alpha^*} f_c(\alpha,\alpha^*), \quad \text{etc.}$$

- The integral over the complex plane $\int d^2\alpha = \int d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha) = \int \frac{dx\,dp}{2}$

- A delta function on the complex plane $\delta^{(2)}(\alpha-\beta) = \delta(\mathrm{Re}(\alpha) - \mathrm{Im}(\beta))\,\delta(\mathrm{Im}(\alpha) - \mathrm{Im}(\beta))$

$\left(\alpha = \frac{x+ip}{\sqrt{2}},\ \beta = \frac{x'+ip'}{\sqrt{2}}\right)$ $\qquad = 2\,\delta(x-x')\,\delta(p-p')$

We are also interested in the 2D Fourier Transform. Recall for one variable

$$\tilde{f}(k) \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx} f(x), \qquad f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \tilde{f}(k)\, e^{ikx}$$

Here we are interested in the Fourier transform of a function on phase space $f(X,P)$. We consider a "reciprical space" X', P'

$$\tilde{f}(X', P') = \int \frac{dX\,dP}{2\pi}\, e^{-i(PX' - XP')} f(X,P), \qquad f(X,P) = \int \frac{dX'\,dP'}{2\pi}\, e^{-i(P'X - X'P)} \tilde{f}(X',P')$$

or in complex form

$$\tilde{f}_c(\beta) = \int \frac{d^2\alpha}{\pi}\, e^{\beta\alpha^* - \beta^*\alpha} f_c(\alpha), \qquad f_c(\alpha) = \int \frac{d^2\beta}{\pi}\, e^{\alpha\beta^* - \alpha^*\beta}\, \tilde{f}_c(\beta)$$

Note: Here P' is reciprical to X and X' is recripical to P

The 2D plane waves span the space of functions on phase space. They are an orthonormal set of functions on the complex plane.

Let $g^{X_0,P_0}(X,P) = e^{i(PX_0 - XP_0)}$ $\qquad g_c^{\alpha_0} = e^{\alpha\alpha_0^* - \alpha^*\alpha_0}$

$$\Rightarrow \langle g^{X_1,P_1} | g^{X_0,P_0} \rangle = \int dX\,dP\, e^{iP(X_0 - X_1)}\, e^{-iX(P_0 - P_1)} = (2\pi)^2\, \delta(X_0 - X_1)\, \delta(P_0 - P_1)$$

$$\Rightarrow \int d^2\alpha\, e^{\alpha(\alpha_0 - \alpha_1)^* - \alpha^*(\alpha_0 - \alpha_1)} = \pi^2 \delta^{(2)}(\alpha_0 - \alpha_1)$$

or $\displaystyle \int \frac{dX\,dP}{2\pi}\, e^{i(P(X_0 - X_1) - X(P_0 - P_1))} = 2\pi\, \delta(X_0 - X_1)\, \delta(P_0 - P_1)$

$$\int \frac{d^2\alpha}{\pi}\, e^{\alpha(\alpha_0 - \alpha_1)^* - \alpha^*(\alpha_0 - \alpha_1)} = \pi\, \delta^{(2)}(\alpha_0 - \alpha_1)$$

These completeness and orthogonality relations will be the foundation for our generalization to operator functions of $\hat{X}$ and $\hat{P}$ or equivalently $\hat{a}$ and $\hat{a}^\dagger$