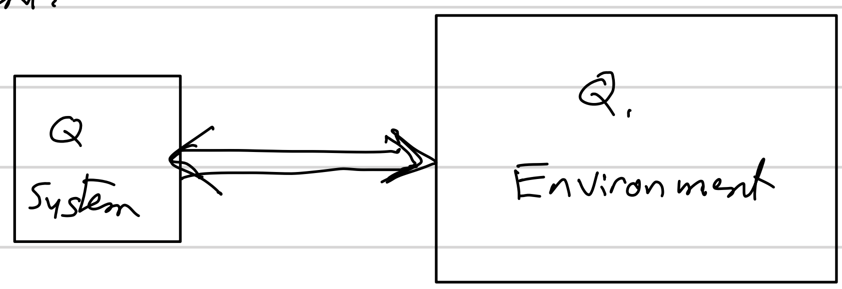
Physics 581: Open Quantum Systems
Lecture 8: Quantum Channels

Our description of an open quantum system divides the universe into subsystems, the quantum system of interest and the rest of the universe, which we call the "environment."



If this is all there is in the universe, then the overall joint evolution of system + environment is unitary. The state of the system alone is obtained by tracing out the enviroment. Thus, given the intial joint state $\hat{\rho}_{SE}(0)$, at a later time

$$\hat{\rho}_S(t) = \mathrm{Tr}_E\left(\hat{U}_{SE}(t)\,\hat{\rho}_{SE}(0)\,\hat{U}^{\dagger}_{SE}(t)\right)$$

Note: The same would be true for any bipartite (and otherwise isolated) system. Whether we call the ancillary system to be the "environment" will be a more subtle question, regarding irreversibility and entropy.

Of particular interest is when the system and the environment are uncorrelated at the initial time: $\hat{\rho}_{SE}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$. In particular let us take the state of the environment to be a pure state $\hat{\rho}_E(0) = |0\rangle_E\langle 0|$, where $|0\rangle_E$ is known as the "fiducial state." This may seem an odd choice for the environment as we probability have very little information about E, so we should generically say it is a mixed, e.g. thermal state. We are doing this here for two reasons. Firstly, this can be a good approximation to a physical situation when the temperature of the enviroment is low compared to the energy of excitations of the system, e.g. the electromagnetic blackbody at room temperature compared to optical excitations. In that case, we E is approximately the vacuum $|0\rangle_E$

Give $\hat{\rho}_{SE}(0) = \hat{\rho}_S(0) \otimes |0\rangle\langle 0|$

basis for the enviroment

$$\hat{\rho}_S(t) = \mathrm{Tr}_E\left(\hat{U}_{SE}(t)\,\hat{\rho}_S(0) \otimes |0\rangle_E\langle 0|\,\hat{U}^{\dagger}_{SE}(t)\right) = \sum_{\mu_E} {}_E\langle\mu|\,\hat{U}_{SE}(t)\,\hat{\rho}_S(0) \otimes |0\rangle_E\langle 0|\,\hat{U}^{\dagger}_{SE}(t)\,|\mu\rangle_E$$

$$\Rightarrow \boxed{\hat{\rho}_S(t) = \sum_{\mu} \hat{M}_{\mu}(t)\,\hat{\rho}_S(0)\,\hat{M}^{\dagger}_{\mu}(t) \quad \text{where} \quad \hat{M}_{\mu}(t) = \underbrace{{}_E\langle\mu|\,\hat{U}_{SE}(t)\,|\mu\rangle_E}_{\text{partial matrix element}}}$$

$\{\hat{M}_{\mu}\}$ are know as the "Kraus operators" or the "measurement operators". The latter is more descriptive, because as we will see, the open quantum system dynamics is intimately related to the measurement process. The Kraus operators act on the system degrees of freedom, as we have traced out E.

Note, that when the dynamics is separable $\hat{U}_{SE}(t) = \hat{U}_S(t) \otimes \hat{U}_E(t)$

$\hat{M}_{\mu}(t) = {}_E\langle\mu|\,\hat{U}_E(t)\,|0\rangle_E\,\hat{U}_S(t)$ and $\hat{\rho}_S(t) = \underbrace{\sum_{\mu} {}_E\langle 0|\,\hat{U}_{SE}|\mu\rangle_E\langle\mu|\,\hat{U}^{\dagger}_{SE}\,|0\rangle_E}_{\hat{1}_S}\,\hat{U}_S(t)\,\hat{\rho}_S(0)\,\hat{U}^{\dagger}_S(t)$

That is, $\hat{\rho}_S(t)$ evolves unitarily $\rightarrow$ there is one Kraus operator. Contrarily, if the dynamics of S+E is <u>entangling</u>, then the evolution of S is <u>not unitary</u>. For the model we have chosen this is a <u>Kraus map</u> $\equiv$ <u>Quantum channel</u>.

Some properties of the Kraus map:

- Takes physical states to physical states, $\hat{\rho}_S(t) = \hat{\rho}^{\dagger}_S(t)$, $\hat{\rho}_S(t) \geq 0$
  - This follows as $\hat{\rho}_S(t) = \mathrm{Tr}_E(\hat{\rho}_{SE}(t))$ $\leftarrow$ physical state

- $\mathrm{Tr}(\hat{\rho}_S(t)) = \mathrm{Tr}(\hat{\rho}_S(0))$ (Trace preserving)

Proof $\quad \mathrm{Tr}(\hat{\rho}_S(t)) = \mathrm{Tr}\left(\sum_{\mu} \hat{M}_{\mu}(t)\,\hat{\rho}_S(0)\,\hat{M}^{\dagger}_{\mu}(t)\right) = \mathrm{Tr}\left(\sum_{\mu} \hat{M}^{\dagger}_{\mu}\hat{M}_{\mu}\,\hat{\rho}_S(0)\right)$

But $\sum_{\mu} \hat{M}^{\dagger}_{\mu}\hat{M}_{\mu} = \sum_{\mu} {}_E\langle 0|\,\hat{U}^{\dagger}_{SE}\,|\mu\rangle_E\langle\mu|\,\hat{U}_{SE}\,|0\rangle = {}_E\langle 0|\,\hat{U}^{\dagger}_{SE}\hat{U}_{SE}\,|0\rangle_E = \hat{1}_S$

$$\Rightarrow \mathrm{Tr}(\hat{\rho}_S(t)) = \mathrm{Tr}(\hat{\rho}_S(0))$$

To examine the role of entanglement more, consider a an initial pure state of the system so, $|\Psi\rangle_{SE} = |\psi\rangle_S \otimes |0\rangle_E$

$$\Rightarrow \hat{U}_{SE}(t)|\psi\rangle_S \otimes |0\rangle_E = \sum_\mu \langle \mu_E|\hat{U}_{SE}|0\rangle_E |\psi\rangle_S \otimes |\mu_E\rangle = \sum_\mu \hat{M}_\mu |\psi\rangle_S \otimes |\mu\rangle_E$$

This is known the unitary representation of quantum channel.

Note: $\{\hat{M}_\mu\}$ are general not unitary, so $\hat{M}_\mu|\psi\rangle_S$ is not the same norm as $|\psi_S\rangle$

Renormalizing $|\psi_\mu\rangle_S \equiv \dfrac{\hat{M}_\mu|\psi\rangle_S}{\|\hat{M}_\mu|\psi\rangle_S\|} \equiv \dfrac{\hat{M}_\mu|\psi_S\rangle}{\sqrt{p_\mu}}$,

$$p_\mu \equiv \|\hat{M}_\mu|\psi\rangle_S\|^2 = \langle\psi_S|\hat{M}_\mu^\dagger\hat{M}_\mu|\psi_S\rangle, \quad \sum_\mu p_\mu = 1$$

$$\Rightarrow \boxed{\begin{aligned} &\hat{U}_{SE}(t)|\psi\rangle_S \otimes |0\rangle_E = \sum_\mu \sqrt{p_\mu}\, |\psi_\mu\rangle_S \otimes |\mu\rangle_E \quad \text{Entangled state!} \\ &\hat{\rho}_S(t) = \mathrm{Tr}_E\left(|\Psi(t)\rangle_{SE}\langle\Psi(t)|\right) = \sum_\mu{}' p_\mu(t)\, |\psi_\mu(t)\rangle_S\langle\psi_\mu(t)| \;:\; \text{mixed state}\end{aligned}}$$

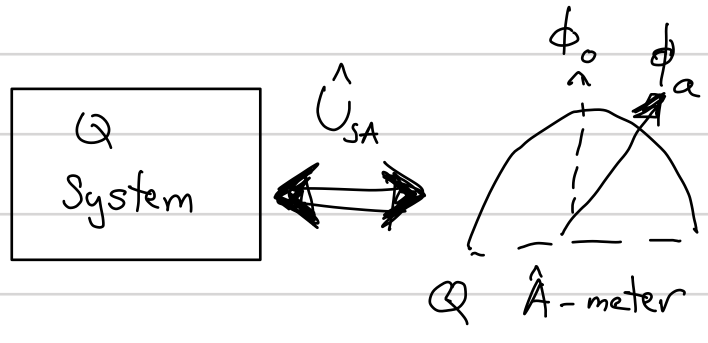
The act of the quantum channel is generally to <u>take a pure state to a mixed state</u>.

The system becomes mixed because it become entangled with the environment. The environment has a "record" of which state the system is in. If the environment is in state $|\mu\rangle_E$, then the system is in $|\psi_\mu(t)\rangle_S$. However, we don't have access to that record. We have incomplete information (it's hidden in the environment). Thus, we must average over μ, leaving a mixed state.

The "record" in the environment, can be considered a kind of measurement. The relationship between measurement and open quantum systems dynamics is deep and critical to our modern understanding of quantum mechanics.

Kraus maps and the measurement model

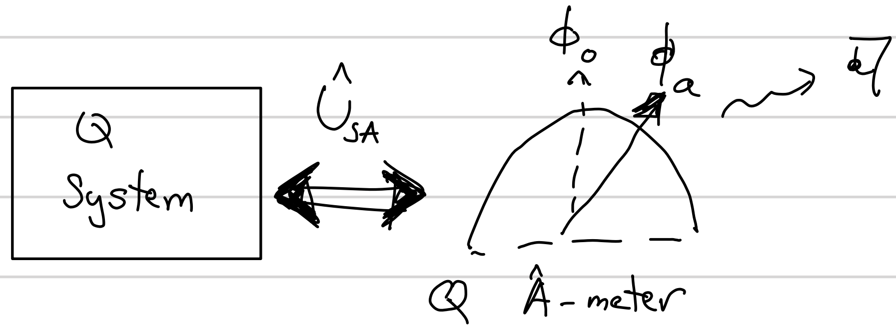
Von Neumann proposed a quantum model for measurement. A quantum "meter" or ancilla is coupled by a quantum system so that the position of the meter is correlated with the value of the observable to be measured.



The system and meter become entangled, and the meter has a record of the state of the system

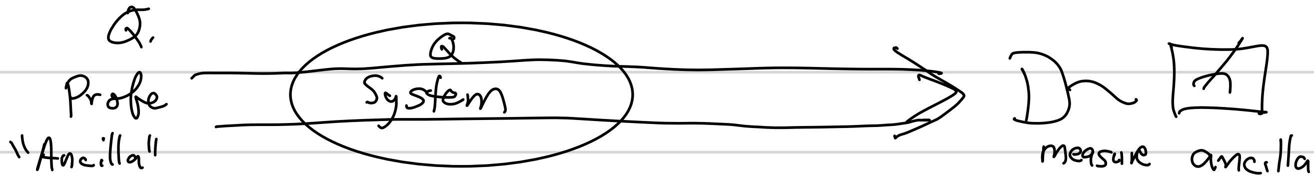
$$\hat{U}_{SA}\,|\psi\rangle_S \otimes \underset{\substack{\uparrow\\ \text{fiducial state}\\ \text{of "pointer"}}}{|\phi_0\rangle_A} = \hat{U}_{SA} \sum_a c_a\,|a\rangle_S \otimes |\phi_0\rangle_A = \sum_a c_a\,|a\rangle_S \otimes \underset{\substack{\uparrow\\ \text{pointer state correlated}\\ \text{with eigenvalue } a}}{|\phi_a\rangle_A}$$

This, of course, doesn't solve the "measurement problem!" Nothing has "collapsed" and we don't see how we ever observe one outcome. To do that an observer must "look at" (measure) the meter!



Depending on how the observer measures the meter determines how the observer learns information about the quantum system. We will study this in more detail later, and in homework. Regardless of its implications in the foundations of quantum mechanics, it is an essential part of our understanding of how measurement actually is carried out in the lab!

Typically we measure a quantum system by probing it, and the measuring the probe



Returning thus to the quantum channel,

Suppose we have access the degrees of freedom of the environment, and can measure it. What have we learned about the system? For simplicity, let's take the system to be in a pure state $|\psi\rangle_S$. After interaction: $|\Psi\rangle_{SE} = \hat{U}_{SE} |\psi\rangle_S \otimes |0\rangle_E$. If we measure the environment in the basis $\{|\mu_E\rangle\}$, the post-measurement state of the system is

$$|\psi_\mu\rangle_S = \frac{{}_E\langle \mu | \Psi_{SE}\rangle}{\| {}_E\langle \mu | \Psi_{SE}\rangle \|} = \frac{{}_E\langle \mu | \hat{U}_{SE} | 0\rangle_E |\psi\rangle_S}{\| \quad \|} = \frac{\hat{M}_\mu |\psi\rangle_S}{\| \hat{M}_\mu |\psi\rangle_S \|}$$

This is quantum Bayes rule. Our "prior" state assignment is $|\psi\rangle_S$. The system and the enviroment become entangled. If we observe the environment in state $|\mu\rangle_E$, we update our state assignment of the system to $|\psi_\mu\rangle_S$ by applying the Kraus operator $\hat{M}_\mu = {}_E\langle \mu | \hat{U}_{SE} | 0\rangle_E$ to the system and the renormalizing.

What is the probability of obtaining this measurement result? According to the Born rule

$$p_\mu = \mathrm{Tr}_{SE}\left(|\Psi_{SE}\rangle\langle\Psi_{SE}|\ |\mu\rangle_E\langle\mu|\right) = \mathrm{Tr}_S\left({}_E\langle\mu|\Psi_{SE}\rangle\langle\Psi_{SE}|\mu\rangle_E\right)$$

$$= \mathrm{Tr}_S\left(\hat{M}_\mu|\psi\rangle_S\langle\psi|\hat{M}_\mu^\dagger\right) = \mathrm{Tr}_S\left(\hat{M}_\mu^\dagger\hat{M}_\mu\ |\psi\rangle_S\langle\psi|\right) = \mathrm{Tr}\left(\hat{E}_\mu\hat{\rho}_S\right)$$

where $\hat{\rho}_S = |\psi\rangle_S\langle\psi|$ $\quad \hat{E}_\mu = \hat{M}_\mu^\dagger M_\mu \quad \sum_\mu \hat{E}_\mu = \sum_\mu \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{\mathbb{1}}_S$

The environment has a "measurement record" of the system. If we look for that record in, e.g., the basis $\{|\mu\rangle_E\}$ we perform of POVM on the system. Two key ingredients:

- Probability of measurement outcome $p_\mu = \mathrm{tr}(\hat{E}_\mu\hat{\rho}_S)$, $\underbrace{\hat{E}_\mu}_{\text{POVM element}} = \hat{M}_\mu^\dagger\hat{M}_\mu$
- Post-measurement state: $|\psi_\mu\rangle = \dfrac{\hat{M}_\mu|\psi\rangle_S}{\sqrt{p_\mu}}$, generally $\hat{\rho}_S|_\mu = \dfrac{\hat{M}_\mu\hat{\rho}_S\hat{M}_\mu^\dagger}{p_\mu}$

The Kraus operators tell the whole story of generalized measurements (POVM). That why they are as called the "measurement operators." They determine the probability of measurement outcomes through the Born rule with POVM elements $\{\hat{E}_\mu = \hat{M}_\mu^\dagger\hat{M}_\mu\}$, and the post-measurement state through quantum Bayes rule,

$$\hat{\rho}_\mu = \frac{\hat{M}_\mu\hat{\rho}_S\hat{M}_\mu^\dagger}{p_\mu}$$

A description of a quantum channel written in term of Kraus map is thus known as a measurement model. The quantum channel can thus be thought of as the environment making a measurement on the system, but not telling us the result. We thus average

$$\hat{\rho}_S^{\,out} = \sum_\mu p_\mu\hat{\rho}_\mu = \sum_\mu \hat{M}_\mu\hat{\rho}\hat{M}_\mu^\dagger$$

Quantum Channels and Kraus Operator Ambiguity

We have written the quantum channel as a Kraus map. For a given input pure state $|\psi^{in}\rangle_S$

$$\hat{\rho}_S^{out} = \sum_\mu{}' \hat{M}_\mu |\psi^{in}\rangle_S\langle\psi^{in}| \hat{M}_\mu^\dagger = \sum_\mu{}' p_\mu |\psi_\mu\rangle_S\langle\psi_\mu|$$

However, as we have studied, the ensemble decomposition is not unique.

let $|\tilde{\psi}_\mu\rangle_S = \sqrt{p_\mu}\,|\psi_\mu\rangle_S = \hat{M}_\mu|\psi\rangle_S$

We know $\hat{\rho}_S^{out} = \sum_\mu^{\mu_{max}}{}' p_\mu |\psi_\mu\rangle_S\langle\psi_\mu| = \sum_\nu^{\nu_{max}} q_\nu |\phi_\nu\rangle_S\langle\phi_\nu|$

iff $|\tilde{\phi}_\nu\rangle = \sum_{\mu=1}^{\mu_{max}} U_{\nu\mu}|\tilde{\psi}_\mu\rangle$ with $|\tilde{\phi}_\nu\rangle = \sqrt{q_\nu}\,|\phi_\nu\rangle$ and $U_{\mu\nu}$ = isometry matrix

Thus $\hat{\rho}_S^{out} = \sum_\nu \sum_{\mu\mu'} U_{\nu\mu}|\tilde{\psi}_\mu\rangle\langle\tilde{\psi}_{\mu'}| U_{\nu\mu'}^\dagger = \sum_\nu \left(\sum_\mu U_{\nu\mu}\hat{M}_\mu\right)|\psi\rangle_S\langle\psi|\left(\sum_{\mu'} U_{\nu\mu'}\hat{M}_{\mu'}\right)^\dagger$

$$\Rightarrow \hat{\rho}_S^{out} = \sum_\nu \hat{N}_\nu |\psi\rangle_S\langle\psi| \hat{N}_\nu^\dagger$$

$$\boxed{\text{Thus, the set of Kraus operators, } \{\hat{M}_\mu\} \text{ and } \{\hat{N}_\nu\} \text{ describe the } \underline{\text{same quantum channel}} \text{ if } \hat{N}_\nu = \sum_\mu U_{\nu\mu}\hat{M}_\mu \quad U_{\nu\mu} = \text{isometry}}$$

We can understand this ambiguity from the point of view of the measurement model. There is more than one kind of measurement record stored in the environment. A given set of Kraus operators corresponds to a particular record we could observe, e.g. $\{\hat{M}_\mu\}$ to $|\mu\rangle_E$ and $\{\hat{N}_\nu\}$ to $|\nu\rangle_E$. But since we didn't obtain this record in the open quantum systems dynamics, the state of system cannot depend on which record we <u>could</u> have done, <u>because we didn't do it!</u> The average over possible measurement outcomes, weighted by their probability is the same quantum channel.

Quantum Channel as a Quantum Map

The input-output relation of a quantum channel can be written as a "quantum map" $\hat{\rho}^{out} = \mathcal{E}(\hat{\rho}^{in})$. $\mathcal{E}$ is a "super operator"

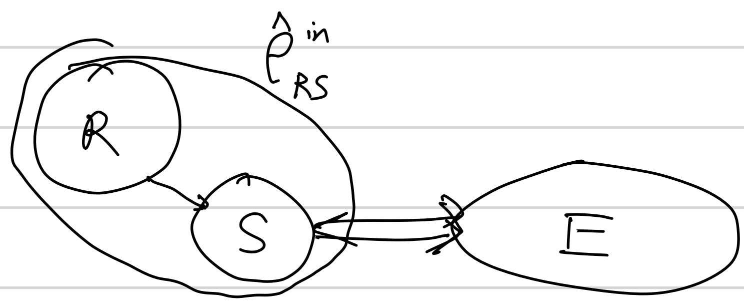
In vectorized notation, $|\hat{\rho}^{out}) = \mathcal{E}|\hat{\rho}^{in})$ $\quad \mathcal{E} = d^2 \times d^2$ matrix

For a quantum channel defined by a Kraus map $\mathcal{E}(\hat{\rho}) = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger$

Satifies:

1) Linearity $\mathcal{E}(c_1\hat{\rho}_1 + c_2\hat{\rho}_2) = c_1\,\mathcal{E}(\hat{\rho}_1) + c_2\,\mathcal{E}(\hat{\rho}_2)$
2) "Positive" $\mathcal{E}(\hat{\rho}) \geq 0$ if $\hat{\rho} \geq 0$
3) Trace preserving: $Tr(\mathcal{E}(\hat{\rho})) = Tr(\hat{\rho})$

It also satisfies another property known as <u>Complete positivity.</u>
Suppose the is some other subsystem (call it the "refrence" subsystem) which might be entangled to the system S. If the map only acts of S, but not R, it should be physical state



$$\hat{\rho}^{out}_S = \mathcal{E}(\hat{\rho}^{in}_S) = \mathcal{E}(Tr_R(\hat{\rho}^{in}_{RS})) = Tr_R\,\tilde{\mathcal{E}}(\hat{\rho}_{RS}) \geq 0 \quad \text{where } \tilde{\mathcal{E}} = \mathcal{I}_R \otimes \mathcal{E}$$

<u>Completely positive</u>: $\tilde{\mathcal{E}}(\hat{\rho}_{RS}) \geq 0$ For all reference systems R

Not every positive map is completely positive, e.g. partial transpose

$$\mathcal{E}_T(\hat{\rho}_S) = \hat{\rho}_S^T \geq 0 \qquad \tilde{\mathcal{E}}_T(\hat{\rho}_R \otimes \hat{\rho}_S) \equiv \hat{\rho}_R \otimes \hat{\rho}_S^T \geq 0$$

But $\tilde{\mathcal{E}}_T(\hat{\rho}_{RS}) \leq 0$ for some entangled states

If $\mathcal{E}$ is a quantum channel (Kraus map) then so is $\tilde{\mathcal{E}}$

$$\tilde{\mathcal{E}}(\hat{\rho}_{RS}) = \mathrm{Tr}_E\left(\hat{U}_{RSE}\,\hat{\rho}_{RS} \otimes |0\rangle_E\langle 0|\,\hat{U}^\dagger_{RSE}\right) = \mathrm{Tr}\left(\hat{\mathbb{1}}_R \otimes \hat{U}_{SE}\,\hat{\rho}_{RS} \otimes |0\rangle_E\langle 0|\,\hat{U}^\dagger_{SE} \otimes \hat{\mathbb{1}}_R\right)$$

$$= \sum_\mu \left(\hat{\mathbb{1}}_R \otimes \hat{M}^\dagger_\mu\right) \hat{\rho}_S \left(\hat{M}^\dagger_\mu \otimes \hat{\mathbb{1}}_R\right) \geq 0$$

Thus any Kraus map is <u>Completely Positive</u>. The converse is true

- A map $\mathcal{E}(\rho)$ is Completely positive (CP-map) iff it is a Kraus map.
- A CP Map can be defined by a "measurement model"

$$\mathcal{E}(\rho) = \mathrm{Tr}_E\left(\hat{U}_{SE}\,\hat{\rho} \otimes |0\rangle_E\langle 0|\,\hat{U}^\dagger_{SE}\right) \quad \text{for some } |0\rangle_E \text{ and } \hat{U}_{SE}$$

We won't prove this here.

<u>Generalized Quantum Operation</u>

We have considered the case of open system dynamics in which we trace over the environment and retain no information stored there. In that case, with no other measurement, $\sum_\mu \hat{M}^\dagger_\mu \hat{M}_\mu = \hat{\mathbb{1}}$, and the CP-map is trace preserving this is sometimes known as a Completely positive trace preserving map (CPTP map). A more general quantum operation coorresponds to the case the we retain some information, but trace over other missing information. In that case $\sum_{\mu=1}^{m \leq \mu_{max}} \hat{M}^\dagger_\mu \hat{M}_\mu < \hat{\mathbb{1}}$, and the map is trace decreasing. We then have quantum Bayes rule

$$\hat{\rho}^{out} = \frac{\sum_\mu^m \hat{M}_\mu \hat{\rho} \hat{M}^\dagger_\mu}{\sum_\mu \mathrm{Tr}(\hat{M}^\dagger_\mu \hat{M}_\mu \hat{\rho})}$$

In addition, the map will be non-trace preserving in we project the state onto a subspace of the system. This is equivalent to measuring a part of the system.

Quantum Channels in the Heisenberg picture

For unitary dynamics

Schrödinger picture: $\hat{\rho}(t) = \hat{U}(t)\,\hat{\rho}\,\hat{U}^\dagger(t)$ $\qquad\qquad$ $\hat{A}(t) = \hat{A}(0) \equiv \hat{A}$

Heisenberg picture: $\hat{\rho}(t) = \hat{\rho}(0) \equiv \hat{\rho}$ $\qquad\qquad$ $\hat{A}(t) = \hat{U}^\dagger(t)\,\hat{A}\,\hat{U}(t)$

Then $\langle \hat{A} \rangle(t) = \mathrm{Tr}(\underbrace{\hat{\rho}(t)\,\hat{A}}_{S.P.}) = \mathrm{Tr}(\underbrace{\hat{\rho}\,\hat{A}(t)}_{H.P.})$ $\qquad$ or any POVM element

For a general quantum channel:

$$\hat{\rho}(t) = \mathcal{E}(t)[\hat{\rho}(0)] = \sum_\mu \hat{M}_\mu^\dagger(t)\,\hat{\rho}\,\hat{M}_\mu(t) \quad : \quad \text{Vectorized} \qquad |\hat{\rho}(t)) = \mathcal{E}(t)\,|\hat{\rho}(0))$$

$$\langle \hat{A} \rangle(t) = \mathrm{Tr}(\hat{A}\,\hat{\rho}(t)) = (\hat{A}|\,\mathcal{E}(t)\,|\hat{\rho}) = (\hat{A}(t)|\,\hat{\rho})$$

where $\quad (\hat{A}(t)| \equiv (\hat{A}|\,\mathcal{E}(t) \Rightarrow \quad |\hat{A}(t)) = \mathcal{E}^\dagger(t)\,|\hat{A})$

$\mathcal{E}^\dagger(t)$ is known as the "adjoint map" (the adjunct of the superoperator). Using the Kraus representation

$$\mathrm{Tr}(\hat{A}\,\hat{\rho}(t)) = \sum_\mu \mathrm{Tr}(A\,\hat{M}_\mu(t)\,\hat{\rho}\,\hat{M}_\mu^\dagger) = \mathrm{Tr}\left(\sum_\mu \hat{M}_\mu^\dagger\,\hat{A}\,\hat{M}_\mu\,\hat{\rho}\right) = \mathrm{Tr}(\hat{A}(t)\,\hat{\rho})$$

$$\Rightarrow \boxed{\mathcal{E}^\dagger(t)\,[\hat{A}] = \sum_\mu \hat{M}_\mu^\dagger(t)\,\hat{A}\,\hat{M}_\mu(t) \qquad \text{(adjoint map)}}$$

Note: For a trace-preserving quantum channel

$$\hat{\mathcal{E}}^\dagger(t)[\hat{\mathbb{1}}] = \sum_\mu \hat{M}_\mu^\dagger(t)\,\hat{M}_\mu(t) = \hat{\mathbb{1}} \Rightarrow \text{the adjoint map is "unital"}$$

Note, in contrast $\hat{\mathcal{E}}(t)[\hat{\mathbb{1}}] = \sum_\mu \hat{M}_\mu(t)\,\hat{M}_\mu^\dagger$. This is not always $= \hat{\mathbb{1}}$. Channels for which $\hat{\mathcal{E}}[\hat{\mathbb{1}}] = \hat{\mathbb{1}}$ are called <u>unital channels</u>. They map the max mixed state $\hat{\rho} = \frac{1}{d}\hat{\mathbb{1}}$ to itself. A non-unital channel will act to purify the state (i.e. cool it). This is perfectly fine!

Examples of Quantum Channels for qubits

Let us consider the quantum system to be qubit, $|\uparrow_z\rangle \equiv |0\rangle$ $|\downarrow_z\rangle \equiv |1\rangle$.
There are three quantum channels of common interest. We will study soon their physical origin.

- **Depolarizing Channel:**

Unitary representation: $\hat{U}_{depol} |\psi\rangle_S \otimes |0\rangle_E = \sqrt{1-p}\, |\psi\rangle_S \otimes |0\rangle_E$

$$+ \sqrt{\tfrac{p}{3}}\, \hat{\sigma}_x |\psi\rangle_S \otimes |1\rangle_E + \sqrt{\tfrac{p}{3}}\, \hat{\sigma}_y |\psi_S\rangle \otimes |2\rangle_E + \sqrt{\tfrac{p}{3}}\, \hat{\sigma}_z |\psi_S\rangle \otimes |3\rangle_E$$

The depolarizing channel can interpreted at with probability $1-p$, nothing happens, and with probability p either $\hat{\sigma}_x$, $\hat{\sigma}_y$, or $\hat{\sigma}_z$ acts on system with equal likelihood.

The Kraus operators: $\hat{M}_\alpha = {}_E\langle \alpha | \hat{U}_{depor} |0\rangle_E$ $\quad \alpha = 0, 1, 2, 3$

$\Rightarrow \hat{M}_0 = \sqrt{1-p}\, \mathbb{1}$, $\hat{M}_j = \sqrt{\tfrac{p}{3}}\, \hat{\sigma}_j$ $\left(j = \underset{x,y,z}{1,2,3} \right)$

This is trace preserving $\displaystyle\sum_{\alpha=0}^{3} \hat{M}_\alpha^\dagger \hat{M}_\alpha = \left[(1-p) + 3\tfrac{p}{3}\right] \hat{\mathbb{1}} = \hat{\mathbb{1}} \underbrace{= \sum_\alpha \hat{M}_\alpha \hat{M}_\alpha^\dagger}_{\text{unital}}$ Since $\hat{M}_\alpha = \hat{M}_\alpha^\dagger$

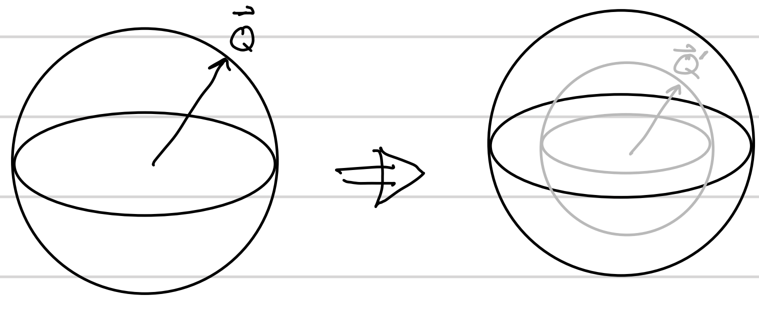
$$\mathcal{E}_{depol}[\hat{\rho}] = (1-p)\hat{\rho} + \underbrace{\tfrac{p}{3}\left(\hat{\sigma}_x \hat{\rho} \hat{\sigma}_x + \hat{\sigma}_y \hat{\rho} \hat{\sigma}_y + \hat{\sigma}_z \hat{\rho} \hat{\sigma}_z \right)}_{\text{"twirl"}}$$

For a general state $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}\cdot\vec{\sigma})$, with some algebra (see homework)

$$\mathcal{E}_{depol}[\hat{\rho}] = \tfrac{1}{2}(\hat{\mathbb{1}} + \vec{Q}'\cdot\hat{\vec{\sigma}}) \qquad \vec{Q}' = \left(1 - \tfrac{4}{3}p\right)\vec{Q}$$

$$= \left(1 - \tfrac{4}{3}p\right)\hat{\rho} + \left(\tfrac{4}{3}p\right)\frac{\hat{\mathbb{1}}}{2}$$ [convex mixture with max mixed state]

The depolarizing channel thus describes an isotropic contraction of Bloch sphere for $p < \frac{3}{4}$.



Note for $p > \frac{3}{4}$ the direction of $\vec{Q}$ reverses. When $p = 1$ $\vec{Q}' = -\frac{1}{3}\vec{Q}$

- <u>Dephasing Channel</u>

Unitary representation: $\hat{U}_{deph} |0\rangle_S \otimes |0\rangle_E = \sqrt{1-p}\, |0\rangle_S \otimes |0\rangle_E + \sqrt{p}\, |0\rangle_S \otimes |1\rangle_E$

$\hat{U}_{deph} |1\rangle_S \otimes |0\rangle_E = \sqrt{1-p}\, |1\rangle_S \otimes |0\rangle_S + \sqrt{p}\, |1\rangle_S \otimes |2\rangle_E$

This is an odd definition, but becomes more clear when we write the Kraus op.

$${}_E\langle 0| \hat{U}_{deph} |0\rangle_S |0\rangle_E = \sqrt{1-p}\, |0\rangle_S \qquad {}_E\langle 0| \hat{U}_{depol} |1\rangle_S \otimes |0_E\rangle = \sqrt{1-p}\, |1\rangle_S$$

$$\Rightarrow \hat{M}_0 = \sqrt{1-p}\, \hat{\mathbb{1}}$$

$${}_E\langle 1| \hat{U}_{deph} |0\rangle_S |0\rangle_E = \sqrt{p}\, |0\rangle_S\,, \quad {}_E\langle 1| \hat{U}_{deph} |1\rangle_S \otimes |0\rangle_E = 0$$

$$\Rightarrow \hat{M}_1 = \sqrt{p}\, |0\rangle\langle 0| = \sqrt{p} \left(\frac{\mathbb{1} + \hat{\sigma}_z}{2} \right)$$

$${}_E\langle 2| \hat{U}_{depol} |0\rangle_S |0\rangle_E = 0\,, \quad {}_E\langle 2| \hat{U}_{deph} |1\rangle_S \otimes |0\rangle_E = \sqrt{p}\, |1_S\rangle$$

$$\Rightarrow \hat{M}_2 = \sqrt{p}\, |1\rangle\langle 1| = \sqrt{p} \left(\frac{\mathbb{1} - \hat{\sigma}_z}{2} \right)$$

$$\mathcal{E}[\hat{\rho}] = \sum_{\alpha=0}^{2} \hat{M}_\alpha \hat{\rho} \hat{M}_\alpha^\dagger = (1 - \tfrac{1}{2}p)\, \hat{\rho} + \frac{p}{2} \hat{\sigma}_z \hat{\rho} \hat{\sigma}_z = \sum_{\beta=0}^{1} \hat{N}_\beta \hat{\rho} \hat{N}_\beta^\dagger$$

<u>Unital</u)

$$\hat{N}_0 = \sqrt{1 - \tfrac{p}{2}}\, \hat{\mathbb{1}}\,, \quad \hat{N}_1 = \sqrt{\tfrac{p}{2}}\, \hat{\sigma}_z$$

$$\mathcal{E} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}$$

The depolarizing channel damps the coherences with change populations

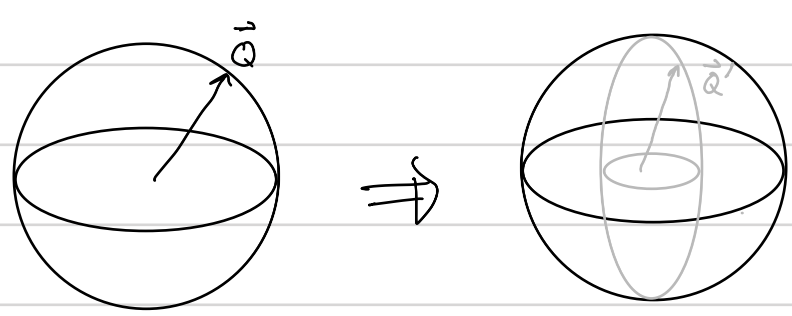
It is a pure "T_2" decay studied earlier

Bloch sphere: $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q} \cdot \hat{\vec{\sigma}})$ $\quad \mathcal{E}[\hat{\rho}] = \frac{1}{2}(\hat{\mathbb{1}} + \vec{Q}' \cdot \hat{\vec{\sigma}})$

$$Q_x' = (1-p)\, Q_x\,, \quad Q_y' = (1-p)\, Q_y\,, \quad Q_z' = Q_z$$

$$u' = (1-p)\, u\,, \quad v' = (1-p)\, v\,, \quad w' = w$$

Bloch Sphere



Amplitude Damping Channel

- Unitary Representation

$$\hat{U}_{amp} |0\rangle_S \otimes |0\rangle_E = \sqrt{1-p}\, |0\rangle_S \otimes |0\rangle_E + \sqrt{p}\, |1\rangle_S \otimes |1\rangle_E$$

$$\hat{U}_{amp} |1\rangle_S \otimes |0\rangle_E = |1\rangle_S \otimes |0\rangle_E$$

The amplitude damping takes the excited state (here $|0\rangle$) to the ground state (here $|1\rangle$) with prob. p, and leaves the ground state alone

- The Kraus operators

$${}_E\langle 0| \hat{U}_{amp} |0\rangle_S \otimes |0\rangle_E = \sqrt{1-p}\, |0\rangle_S \quad , \quad {}_E\langle 0| \hat{U}_{amp} |1\rangle_S \otimes |0\rangle_E = |1\rangle_S$$

$$\Rightarrow \hat{M}_0 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{bmatrix}$$

$${}_E\langle 1| \hat{U}_{amp} |0\rangle_S \otimes |0\rangle_E = \sqrt{p}\, |1\rangle_S \quad , \quad {}_E\langle 1| \hat{U}_{amp} |1\rangle_S \otimes |0\rangle_E = 0$$

$$\Rightarrow \hat{M}_1 = \begin{bmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{bmatrix} = \sqrt{p}\, \hat{\sigma}_-$$

$$\hat{M}_0^\dagger \hat{M}_0 + \hat{M}_1^\dagger \hat{M}_1 = \begin{bmatrix} 1-p & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\hat{M}_0 \hat{M}_0^\dagger + \hat{M}_1 \hat{M}_1^\dagger = \begin{bmatrix} 1-p & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} 1-p & 0 \\ 0 & 1+p \end{bmatrix} \neq \hat{\mathbb{1}}$$ not unital!

$$\mathcal{E}_{amp}[\hat{\rho}] = \begin{bmatrix} (1-p)\rho_{00} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{01} & p\rho_{00} + \rho_{11} \end{bmatrix}$$

"T_1 process" population and coherence decay

Bloch sphere