Physics 581 – Quantum Optics II
Lecture 9 – The Lindblad Master Equation

In the previous lecture, we studied abstract maps on open quantum systems. While this gives us a general framework for describing input-output maps it does not directly describe the physics of the dynamics. For that we need to consider a differential equation and the generator of dynamics.

Consider first Closed Quantum Systems

- General input-output: Unitary evolution $\hat{U}(t+\tau, t)$

$$\hat{\rho}(t+\tau) = \hat{U}(t+\tau, t)\,\hat{\rho}(t)\,\hat{U}^{\dagger}(t+\tau, t)$$

- Generator of dynamics: The Hamiltonian $\hat{H}(t)$

Near unity differential map: $\hat{U}(t+dt; t) = \hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}(t)\,dt$

- Differential evolution: The Schrödinger equation

$$\hat{\rho}(t+dt) = \hat{U}(t+dt; t)\,\hat{\rho}(t)\,\hat{U}^{\dagger}(t+dt; t)$$

$$= \left(\hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}(t)\,dt\right)\hat{\rho}\left(\hat{\mathbb{1}} + \frac{i}{\hbar}\hat{H}(t)\,dt\right) = \hat{\rho}(t) - \frac{i}{\hbar}[\hat{H}(t), \hat{\rho}(t)]\,dt$$

$$\Rightarrow \hat{\rho}(t+dt) = \hat{\rho}(t) - \frac{i}{\hbar}\,ad_{\hat{H}(t)}[\hat{\rho}(t)]\,dt \quad \left(ad_{\hat{A}}[\hat{B}] \equiv [\hat{A}, \hat{B}]\right)$$

$$\text{or} \quad \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\,ad_{\hat{H}(t)}[\hat{\rho}(t)] = -\frac{i}{\hbar}[\hat{H}(t), \hat{\rho}(t)]$$

Equivalently, $\dfrac{\partial \hat{U}}{\partial t} = -\dfrac{i}{\hbar}\hat{H}(t)\,\hat{U} \Rightarrow \hat{U}(t_2, t_1) = \underbrace{\mathcal{T}}_{\text{time ordering}}\left\{e^{-\frac{i}{\hbar}\int_{t_1}^{t_2}\hat{H}(t)\,dt}\right\} = e^{-\frac{i}{\hbar}\hat{H}(t_2 - t_1)}$ if $\hat{H}$ time-independent

Consider now the analogous analysis for <u>Open Quantum Systems</u>

- General input-output: <u>CP Map</u> $\mathcal{A}(t+\tau, t)$

$$\hat{\rho}(t+\tau) = \mathcal{A}(t+\tau; t)[\hat{\rho}(t)]$$

- Generator of dynamics: <u>The "Lindbladian"</u> $\mathcal{L}(t)$

Near unity differential map: $\mathcal{A}(t+dt; t)$.
Conjecture a form: $\mathcal{A}(t+dt; t)[\hat{\rho}(t)] = \hat{\rho}(t) + \underbrace{\mathcal{L}(t)[\hat{\rho}(t)]}dt$

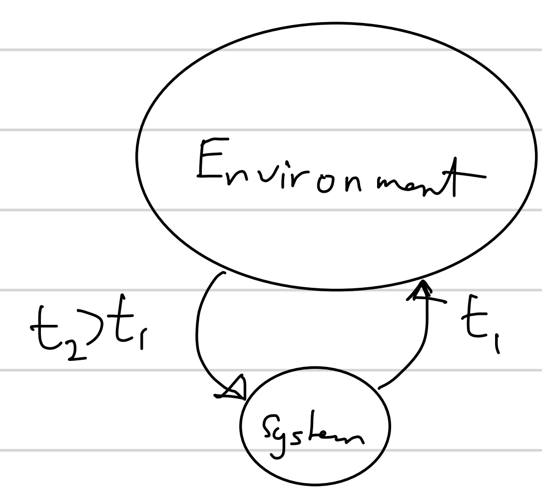
depends only on t and not whole history

this form is only true under the <u>Markoff approximation</u>.
whereby $\hat{\rho}(t+d\tau)$ depends only on $\hat{\rho}(t)$ and not the whole history of $\hat{\rho}$ from $0 \to t$.

Recall, a stochastic process is said to be Markoff if

$$P(x_k t_k | x_{k-1} t_{k-1}, \ldots, x_2 t_2, x_1 t_1) = P(x_k t_k | x_{k-1} t_{k-1}) \Rightarrow \text{No "memory"}$$

In open system dynamics, this can only every be approximate.
The reason is that the total evolution of $\hat{\rho}_S(t)$ depends on the whole history in $0 \to t$, as information flows to the environment and back



- Differential evolution: <u>The Master equation</u>

$$\hat{\rho}(t+dt) = \hat{\rho}(t) + \mathcal{L}(t)[\hat{\rho}(t)]\,dt$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = \mathcal{L}(t)[\hat{\rho}(t)]$$

equivalently $\frac{d\mathcal{A}}{dt} = \mathcal{L}(t) \circ \mathcal{A} \Rightarrow \mathcal{A}[t_2; t_1] = \mathcal{T}\left\{ e^{\int_{t_1}^{t_2} \mathcal{L}(t)\,dt} \right\}$

$\mathcal{A}[t_2; t_1] = e^{(t_2 - t_1)\mathcal{L}}$ when $\mathcal{L}$ time independent

The form of Master equation does not yet contain much information
We can find general form of Lindbladian using the Kraus representation of the differential map

$$\mathcal{A}[t+dt; t)[\hat{\rho}(t)] = \hat{\rho}(t) + \mathcal{L}(t)[\hat{\rho}(t)]\,dt = \sum_{\mu=0}^{M_{max}} \hat{M}_\mu(t, dt)\,\hat{\rho}(t)\,\hat{M}_\mu^\dagger(t, dt)$$

We can take one of the Kraus operators similar to a near-unity unitary operator

$$\hat{M}_0 = \hat{\mathbb{1}} + \hat{G}(t)\, dt$$

All other Kraus operators must be of the form $\hat{M}_\mu = \hat{L}_\mu(t)\sqrt{dt}$
the operators $\{\hat{L}_\mu(t)\}$ are known as the Lindblad operators (also known as the "jump operators," for reasons that will become clear).

We consider here trace preserving maps. This requires $\sum_\mu \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{\mathbb{1}}$

$$\Rightarrow \hat{M}_0^\dagger \hat{M}_0 = \hat{\mathbb{1}} - \sum_{\mu=1}^{M_{max}} \hat{M}_\mu^\dagger \hat{M}_\mu \Rightarrow \hat{\mathbb{1}} + \left(\hat{G}(t) + \hat{G}^\dagger(t)\right)dt = \hat{\mathbb{1}} - \sum_{\mu=1} \hat{L}_\mu^\dagger(t)\hat{L}_\mu(t)\,dt$$

$$\Rightarrow \hat{G} + \hat{G}^\dagger = -\sum_{\mu=1}^{M_{max}} \hat{L}_\mu^\dagger \hat{L}_\mu \Rightarrow \hat{G}(t) = -\frac{1}{2}\sum_{\mu=1}^{M_{max}} \hat{L}_\mu^\dagger \hat{L}_\mu + \hat{A}$$

arbitrary anti-Hermitian

We see from the form of $\hat{M}_0$, $\hat{A}(t) = \frac{-i}{\hbar}\hat{H}(t)$. $\leftarrow$ the Hamiltonian!

We sometimes define $\hat{G}(t) = -\frac{i}{\hbar}\hat{H}_{eff}(t)$

where $\hat{H}_{eff} = \hat{H} - \frac{i\hbar}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu$; this is a familiar form from last semester

$$\text{Thus } \hat{\rho}(t+dt) = \sum_{\mu=0}^{M_{max}} \hat{M}_\mu(dt;t)\,\hat{\rho}(t)\,\hat{M}_\mu^\dagger(dt;t) = \hat{M}_0\hat{\rho}\hat{M}_0^\dagger + \sum_{\mu=1}^{M_{max}} \hat{M}_\mu \hat{\rho}\hat{M}_\mu^\dagger$$

$$= \hat{\rho}(t) - \frac{i}{\hbar}\left(\hat{H}_{eff}(t)\,\hat{\rho}(t) - \hat{\rho}(t)\,\hat{H}_{eff}^\dagger\right)dt + \sum_{\mu=1}^{M_{max}} \hat{L}_\mu(t)\,\hat{\rho}\,\hat{L}_\mu^\dagger(t)\,dt$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = \mathcal{L}(t)[\hat{\rho}(t)] = -\frac{i}{\hbar}[\hat{H}_{eff}(t), \hat{\rho}(t)]^* + \sum_\mu \hat{L}_\mu(t)\,\hat{\rho}(t)\,\hat{L}_\mu^\dagger(t)$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}(t), \hat{\rho}] + \sum_\mu \left\{ -\frac{1}{2}\left(\hat{L}_\mu^\dagger(t)\hat{L}_\mu(t)\,\hat{\rho}(t) + \hat{\rho}(t)\,\hat{L}_\mu^\dagger(t)\hat{L}_\mu(t)\right) + \hat{L}_\mu(t)\hat{\rho}(t)\hat{L}_\mu^\dagger(t) \right]$$

This is our central result – The Master Equation in Lindblad form.
This is the most general form of differential equation for $\hat{\rho}$ in the Markoff approximation that yields a CP map.

To get a basic understanding of the Lindblad operators, consider the rate equations – evolution of populations. Generally, as we will see, the Lindblad operators couples populations with populations and coherences with coherences. The coupling of populations with coherences is achieved by a driving Hamiltonian. Let us consider only the Lindblad dissipative part.

Consider the basis of energy eigenstates $\{|g\rangle\}$, $P_g \equiv \langle g|\hat{\rho}|g\rangle$

$$\Rightarrow \dot{P}_g = \langle g|\frac{d\hat{\rho}}{dt}|g\rangle = -\frac{1}{2}\sum_\mu \left(\langle g|\hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho}|g\rangle + \langle g|\hat{\rho}\hat{L}_\mu^\dagger \hat{L}_\mu|g\rangle\right)$$
$$+\sum_\mu \langle g|\hat{L}_\mu \hat{\rho}\hat{L}_\mu^\dagger|g\rangle$$

In the typical master equations we encounter, the dissipation does not couple population to coherences, so let's just consider diagonal terms $\hat{\rho} = \sum_k P_k |k\rangle\langle k|$

$$\Rightarrow \dot{P}_g = -\gamma_g P_g + \sum_k \gamma_{g\leftarrow k} P_k$$

where $\gamma_{g\leftarrow k} \equiv \sum_\mu |\langle g|\hat{L}_\mu|k\rangle|^2$, $\gamma_g = \sum_\mu \langle g|\hat{L}_\mu^\dagger \hat{L}_\mu|g\rangle = \sum_k \gamma_{k\leftarrow g}$

From this we can make the following interpretations:

- The matrix elements of the Lindblad operators determine the transition rates for population transfer $\gamma^\mu_{g\leftarrow k} = |\langle g|\hat{L}_\mu|k\rangle|^2$. The different $\{\mu\}$ represent different channels for transfer. The total transfer rate is $\sum_\mu \gamma^\mu_{g\leftarrow k}$. Thus the Lindblad operators are also known as the "jump operators."

- $\gamma_g = \sum_k \gamma_{k\leftarrow g}$ is the total decay rate out of g

- Writing $\hat{H}_{eff} = \hat{H} - \frac{i\hbar}{2}\sum \hat{L}_\mu^\dagger \hat{L}_\mu$, the anti-Hermitian part determines the rate of decay out of a given state.

- The term $\sum_\mu \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$ accounts for "re-feeding" of population

- We can see the equation as a standard rate equation by noting that
$\gamma_j = \sum_{k \neq j} \gamma_{k \leftarrow j} + \gamma_{k \leftarrow k}$

$$\Rightarrow \dot{P}_k = \sum_{j \neq k} \left(-\gamma_{k \leftarrow j} P_j + \gamma_{j \leftarrow k} P_k \right)$$

Given the interpretation of the Lindblad jump operators, we can often write down the Master Equation for a particular physical application without a detailed derivation. Consider, for example, the decay of a two-level atom due to its coupling to the electromagnetic vacuum. Physically, we know $\gamma_{g \leftarrow e} = \Gamma$ (the Einstein-A coefficient) and $\gamma_{e \leftarrow g} = 0$.

$\Rightarrow$ We can express the Master Equation using on jump operator

$$|\langle e|\hat{L}|g\rangle|^2 = \Gamma, \quad |\langle g|\hat{L}|e\rangle|^2 = 0 \quad \Rightarrow \quad \hat{L} = \sqrt{\Gamma}\, |g\rangle\langle e| = \sqrt{\Gamma}\, \hat{\sigma}_-$$

$$\Rightarrow \text{Master equation: } \left.\frac{d\hat{\rho}}{dt}\right|_{diss} = -\frac{\Gamma}{2}\left(\hat{\sigma}_- \hat{\sigma}_+ \hat{\rho} + \hat{\rho} \hat{\sigma}_- \hat{\sigma}_+ \right) + \Gamma \hat{\sigma}_- \hat{\rho} \hat{\sigma}_+$$

This is the Master Equation we saw last semester in Lecture 7 in the context of the optical Bloch equations. Note, substituting for $\hat{\sigma}_\pm$

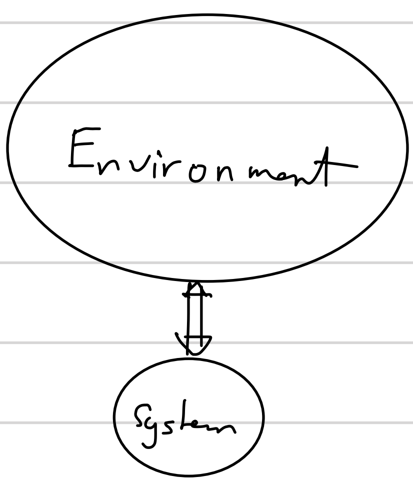
$$\frac{d\hat{\rho}}{dt} = \underbrace{-\frac{i}{\hbar}\left(\hat{H}_{decay} \hat{\rho} - \hat{\rho} \hat{H}_{decay}^\dagger \right)}_{\text{decay out of excited state}} + \underbrace{\Gamma \rho_{ee} |g\rangle\langle g|}_{\text{feeding back in ground state}}$$

$$\hat{H}_{decay} = -i \frac{\hbar}{2}\Gamma |e\rangle\langle e|$$

While in this case, we were able to guess the form of the Master eqn. this is not always true. In addition, we would like to have a deeper physical understanding of the nature of the approximations. We thus turn to a first principles derivation next.

Deriving the Master Equation from first principles

The Lindblad form of the Master Equation followed by the fundamental properties of CP Maps under the Markoff approximation. To explicitly derive the Lindblad operators we must turn to the underlying physics. The basic model is the coupling of the system to the environment. Taken together, these form a "closed system" undergoing unitary evolution



Total Hamiltonian,

$$\hat{H}_{SE} = \hat{H}_S + \hat{H}_E + \hat{H}_{int}$$

We start with the system uncorrelated by the environment $\hat{\rho}_{SE}(0) = \hat{\rho}_S(0)\,\hat{\rho}_E(0)$.
Typically, the environment is taken to be in thermal equilibrium at some temperature T (In quantum optics, this could be the vacuum @ zero temperature).
The effect of the environment is to cause irreversible decay of the system.
We have previously studied this using perturbation theory and the Wigner-Weisskopf approximation. The treatment here is essentially the same, as applied now to the density operator.

As in standard time-dependent perturbation theory, it is convenient go to the interaction picture. The state the evolves according to $\hat{H}_{int}(t)$ (including the free evolution of system and environment).

$$\Rightarrow \frac{d\hat{\rho}_{SE}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_{int}(t),\ \hat{\rho}_{SE}(t)\right]$$

Now, we seek the evolution of the system alone,

$$\hat{\rho}_S(t) = Tr_E(\hat{\rho}_{SE}(t)) \Rightarrow \frac{d\rho_S(t)}{dt} = \frac{-i}{\hbar} Tr_E[\hat{H}_{int}(t), \hat{\rho}_{SE}(t)]$$

Integrating for a short time interval

$$\hat{\rho}_S(t+\Delta t) = \hat{\rho}_S(t) + Tr_E\left\{-\frac{i}{\hbar}\int_t^{t+\Delta t}[\hat{H}_{Int}(t'), \hat{\rho}_{SE}(t')]\,dt'\right\}$$

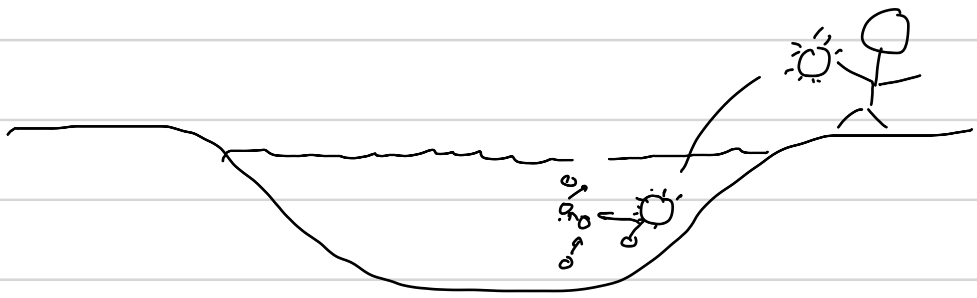
Note, this is not in Markoff form

$$\hat{\rho}_S(t+\Delta t) \neq \hat{\rho}_S(t) + \mathcal{L}(t)[\hat{\rho}_S(t)]\Delta t$$

We can however, get a good approximate solution by coarse graining. There is some characteristic time scale τ_c in which the system and environment are correlated. After this time scale information from the environment is very unlikely to come back to the system. By neglecting the dynamics within the correlation time scale the dynamics becomes irreversible. The general form of the evolution of density operator will be an integro-differential equation $\frac{d\hat{\rho}_S(t)}{dt} = \int_0^t K(t-t')[\hat{\rho}_S(t')]$. The function $K(t-t')$ is the "memory kernal". In the extremely limit of no memory: $K(t-t') \propto \delta(t-t') \Rightarrow \frac{d\hat{\rho}_S(t)}{dt} = \mathcal{L}(t)[\hat{\rho}_S(t)]$

This is known as the Markoff approximation.

We can understand this in a classical analogy of equilibration. Suppose we throw a hot penny into a lake at temperature T.



The water molecules collide with the penny as their motion is correlated in the penny. But very quickly the molecule "rethermalizes" due to many other many other collisions with other molecules in the lake. This rethermalization happens much more quickly than the time it takes for the penny to come to thermal equilibrium. Thus, to study the nonequilibrium dynamics of the system we coarse grain over a time scale Δt

$$\underset{\substack{\uparrow \\ \text{S-E correlation time}}}{\tau_C} \ll \Delta t \ll \underset{\substack{\uparrow \\ \text{system decay time}}}{t_{decay}}$$

Let us return to the formal integration of Schrödinger equation.

$$\hat{\rho}_{SE}(t+\Delta t) = \hat{\rho}_{SE}(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt' \left[\hat{H}_{int}(t'), \hat{\rho}_{SE}(t')\right]$$

$$= \hat{\rho}_{SE}(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt' \left[\hat{H}_{int}(t'), \hat{\rho}_{SE}(t)\right] + \left(\frac{-i}{\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t''} dt'' \left[\hat{H}_{int}(t'), \left[\hat{H}_{int}(t''), \hat{\rho}_{SE}(t'')\right]\right] + \ldots$$

Dyson / Born series

As studied in the Wigner-Weisskopf formulation, we need to go to second-order in the interaction Hamiltonian. Thus we truncated at set $\hat{\rho}_{SE}(t'') \Rightarrow \hat{\rho}_{SE}(t)$ in the last integral. (Born approximation)

$$\Rightarrow \hat{\rho}_S(t+\Delta t) = \mathrm{Tr}_E\left(\hat{\rho}_{SE}(t+\Delta t)\right) = \hat{\rho}_S(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt'\, \mathrm{Tr}_E\left(\left[\hat{H}_{int}(t'), \hat{\rho}_{SE}(t)\right]\right)$$
$$- \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt''\, \mathrm{Tr}_E\left(\left[\hat{H}_{int}(t'), \left[\hat{H}_{int}(t''), \hat{\rho}_{SE}(t)\right]\right]\right)$$

Now comes a key approximation. We can generally write

$$\hat{\rho}_{SE}(t) = \underbrace{\mathrm{Tr}_E\left(\hat{\rho}_{SE}(t)\right) \otimes \mathrm{Tr}_S\left(\hat{\rho}_{SE}(t)\right)}_{\hat{\rho}_S(t) \,\otimes\, \hat{\rho}_E(t)} + \underbrace{\hat{\rho}_{cor}(t)}_{\text{correlations}}$$

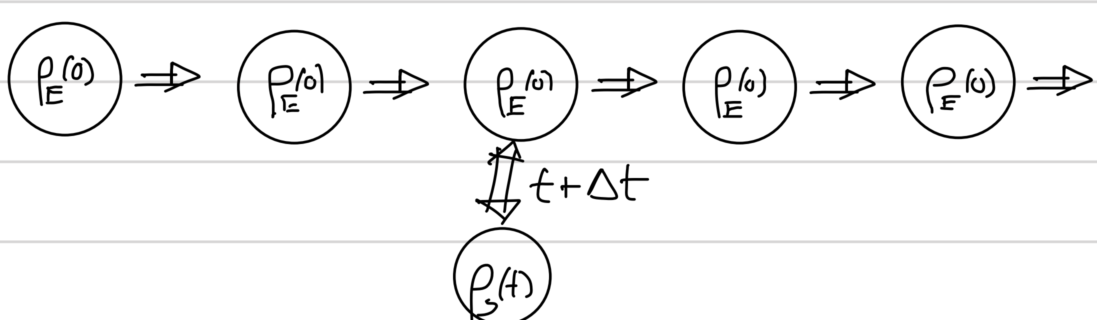
In the Markoff approximation, we can neglect the correlation in the coarse-graining. In this approximation

$$\hat{\rho}_{SE}(t) = \hat{\rho}_S(t) \otimes \hat{\rho}_E(0)$$

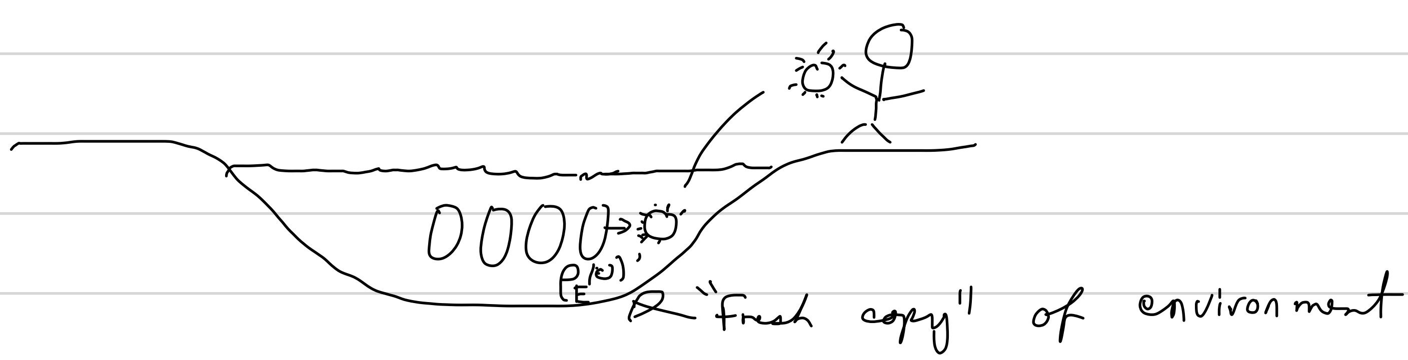
The environment returns to equilibrium after coarse graining

system evolves due to correlations with environment in previous Δt.

Formally, we can think about this as a CP map with a fresh copy of the same environment after each coarse-grained time scale.



This again is intuitive in the classical equilibration problem



$$\hat{\rho}_S(t+\Delta t) \approx \hat{\rho}_S(t) - \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt''\, \mathrm{Tr}_E\left(\left[\hat{H}_{int}(t'), \left[\hat{H}_{int}(t''), \hat{\rho}_S(t)\otimes\hat{\rho}_E(0)\right]\right]\right)$$

$$\Rightarrow \boxed{\begin{aligned}\frac{\Delta\hat{\rho}_S}{\Delta t} \approx -\frac{1}{\hbar^2\Delta t}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \Big\{ &\langle \hat{H}_{int}(t')\hat{H}_{int}(t'')\rangle_E\, \hat{\rho}_S(t) + \hat{\rho}_S(t)\langle \hat{H}_{int}(t'')\hat{H}_{int}(t')\rangle_E \\ &- \langle \hat{H}_{int}(t')\hat{\rho}_S(t)\hat{H}_{int}(t'')\rangle_E - \langle \hat{H}_{int}(t'')\hat{\rho}_S(t)\hat{H}_{int}(t')\rangle_E \Big\}\end{aligned}}$$

This is the first-principles derivation of the Master equation under the Born-Markov approximation. We can interpret this as,

$$\frac{\Delta\hat{\rho}_S}{\Delta t} = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \frac{d\hat{\rho}_S(t')}{dt'},$$

i.e., a coarse-graining of the evolution over the time scale Δt.
We will see this yields the Lindblad form when we give specific examples of the system-environment interaction Hamiltonian.

<u>Example: Equilibration of a two-level atom in a thermal black-body</u>

A canonical example in quantum optics is master equation for a two level atom, where the quantized field serves the role of the "environment". We studied this problem in lecture 14 last semester, when we treated spontaneous emission in the vacuum. We will generalize here to a thermal bath (black body) at temperature T; the vacuum is the special case when $T=0$.

As in Lecture 14, the fundamental Hamiltonian is (in RWA)

$$\hat{H} = \underbrace{\hat{H}_A}_{\text{atom = system}} + \underbrace{\hat{H}_F}_{\text{field = environment}} + \underbrace{\hat{H}_{int}}_{\text{interaction}}$$

$$= \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z + \sum_{\vec{k},\mu}\hbar\omega_k \hat{a}^\dagger_{\vec{k}\mu}\hat{a}_{\vec{k}\mu} + \hbar\sum\left(g_{\vec{k}\mu}\hat{a}_{\vec{k},\mu}\hat{\sigma}_+ + h.c.\right)$$

$$g_{\vec{k}\mu} = \frac{1}{\hbar}\sqrt{\frac{2\pi\hbar\omega_k}{V}}\, \vec{d}_{eg}\cdot\vec{\epsilon}_{\vec{k}\mu}\, e^{i\vec{k}\cdot\vec{r}_A}$$

$\Rightarrow$ In the interaction picture $\hat{H}_{int}(t) = \hbar\hat{F}(t)\hat{\sigma}_+ + \hbar\hat{F}^\dagger(t)\hat{\sigma}_-$

$$\hat{F}(t) \equiv \sum_{\vec{k},\mu} g_{\vec{k}\mu}\hat{a}_{\vec{k}\mu}\, e^{-i(\omega_k-\omega_{eg})t} \equiv \text{"Reservoir noise operator"}$$

The properties of the reservoir field operator.

We take the field to be a thermal state $\hat{\rho}_E = \frac{1}{Z} e^{-\hat{H}_F/k_B T}$

$$\langle \hat{a}_{\vec{k}\mu} \rangle_E = 0 \qquad \langle \hat{a}_{\vec{k}\mu} \hat{a}_{\vec{k}'\mu'} \rangle = \langle \hat{a}^\dagger_{\vec{k}'\mu'} \hat{a}^\dagger_{\vec{k}\mu} \rangle = 0$$

$$\langle \hat{a}^\dagger_{\vec{k}\mu} \hat{a}_{\vec{k}'\mu'} \rangle_E = \bar{n}(\omega_k)\, \delta_{\vec{k}\vec{k}'}\, \delta_{\mu\mu'} \quad , \quad \langle \hat{a}_{\vec{k}\mu} \hat{a}^\dagger_{\vec{k}'\mu'} \rangle = (\bar{n}(\omega_k)+1)\, \delta_{\vec{k}\vec{k}'}\, \delta_{\mu\mu'}$$

$$\Rightarrow \langle \hat{F}(t) \rangle_E = \langle \hat{F}^\dagger(t) \rangle_F = \langle \hat{F}(t_1) \hat{F}(t_2) \rangle_E = \langle \hat{F}^\dagger(t_2) \hat{F}^\dagger(t_1) \rangle = 0$$

$\omega_k - \omega_{eg}$ detuning

$$\langle \hat{F}^\dagger(t_1) \hat{F}(t_2) \rangle = \sum_{\vec{k}\mu} |g_{\vec{k}\mu}|^2 \bar{n}(\omega_k)\, e^{+i\Delta_k(t_1-t_2)} = \int_0^\infty d\omega_k\, \mathcal{D}(\omega_k)\, |\overline{g(\omega_k)}|^2\, \bar{n}(\omega_k)\, e^{+i\Delta_k(t_1-t_2)}$$

$$\langle \hat{F}(t_1) \hat{F}^\dagger(t_2) \rangle = \int_0^\infty d\omega_k\, \mathcal{D}(\omega_k)\, |\overline{g(\omega_k)}|^2 (\bar{n}(\omega_k)+1)\, e^{-i\Delta_k(t_1-t_2)}$$

density of states

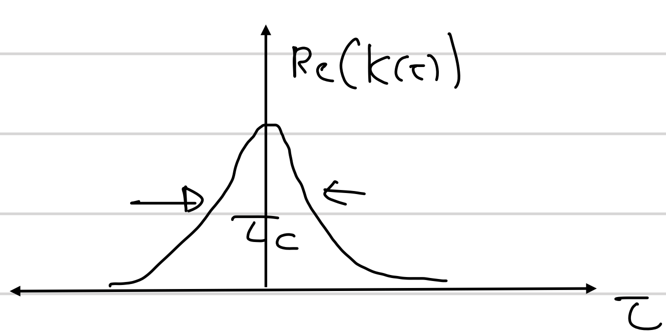
$$\Rightarrow \frac{\Delta \rho_A}{\Delta t} = -\frac{1}{\Delta t} \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \Big\{ (\hat{\sigma}_+ \hat{\sigma}_- \hat{\rho}_A(t) - \hat{\sigma}_- \hat{\rho}_A(t) \hat{\sigma}_+) \langle \hat{F}(t') \hat{F}^\dagger(t'') \rangle_E + (\hat{\sigma}_- \hat{\sigma}_+ \hat{\rho}_A(t) - \hat{\sigma}_+ \hat{\rho}_A(t) \hat{\sigma}_-) \langle \hat{F}^\dagger(t'') \hat{F}(t') \rangle_E + h.c. \Big\}$$

Let us define $K(t_1,t_2) = \langle \hat{F}^\dagger(t_1) \hat{F}(t_2) \rangle_E \equiv$ Reservoir correlation function

- Stationary correlations: $K(t_1,t_2) = K(t_1-t_2) = \langle \hat{F}^\dagger(t_1-t_2) \hat{F}(0) \rangle_E$
- Time-reversal: $K(t_2-t_1) = \langle \hat{F}^\dagger(t_2) \hat{F}(t_1) \rangle_E = K^*(t_1-t_2)$

$$\Rightarrow \mathrm{Re}[K(\tau)] = \mathrm{Re}[K(-\tau)] \quad \text{(symmetric function)}$$

From these properties we can roughly sketch $\mathrm{Re}(K(\tau))$



For sufficiently large τ the correlation function should goes to zero. The characteristic temporal width τ_c is the correlation time. Physically, τ_c sets the time scale in which the environment relaxes back to equilibrium given some excitation from the system. This is the "memory time" of the reservoir.

One can also define a "spectral density" for the reservoir field

$$S(\omega) = \int d\tau\, K(\tau) e^{i\omega\tau}$$

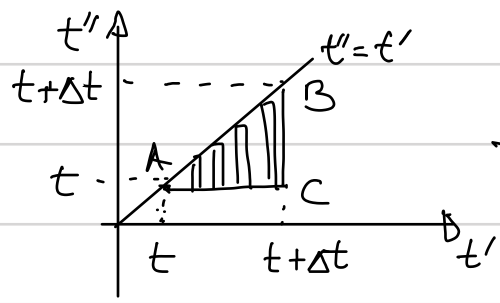
The bandwidth of the reservoir $\Delta\omega_R \sim \frac{1}{\tau_c}$. As $\Delta\omega_R \to \infty$, (i.e., the reservoir is "white noise"), the correlation time $\tau_c \to 0$. The real part of the correlation function is then proportional to a delta function

$$\mathrm{Re}\left(K(\tau)\right) \propto \delta(\tau)$$

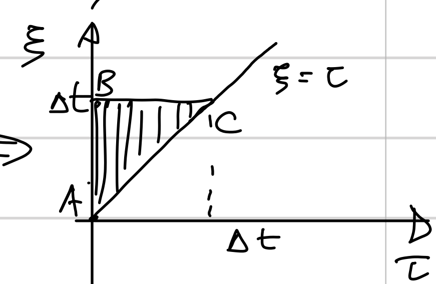
Thus, a "white-noise reservoir" has no memory. This is the Markov approximation

To proceed we make a change of variables: Let $\tau = t' - t''$ $\xi = t' - t$

$$\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \Rightarrow \int_0^{\Delta t} d\tau \int_\tau^{\Delta t} d\xi$$



$\Rightarrow$



Then: $$\Delta\hat{\rho}_A = -\int_0^{\Delta t} d\tau \int_\tau^{\Delta t} d\xi \Big\{ \left(\hat{\sigma}_+ \hat{\sigma}_- \hat{\rho}_A(t) - \hat{\sigma}_- \hat{\rho}_A(t) \hat{\sigma}_+\right) \langle \hat{F}(\tau) \hat{F}^\dagger(0) \rangle + \left(\hat{\sigma}_- \hat{\sigma}_+ \hat{\rho}_A(t) - \hat{\sigma}_+ \hat{\rho}_A(t) \hat{\sigma}_-\right) \langle \hat{F}^\dagger(0) \hat{F}(\tau) \rangle + h.c \Big\}$$

Now, the integrand vanishes for $\tau > \tau_c$. In the Markov approximation $\Delta t \gg \tau_c$, which we take to be $\tau_c \approx 0$. Thus we can take the lower limit of the $d\xi$ integral to zero. Moreover, since $\Delta t \gg \tau_c$, we have take $\Delta t \to \infty$ in the limit of the the integral over τ, giving

$$\frac{\Delta\hat{\rho}_A}{\Delta t} \approx -\int_0^{\infty} d\tau \Big\{ \left(\hat{\sigma}_+ \hat{\sigma}_- \hat{\rho}_A(t) - \hat{\sigma}_- \hat{\rho}_A(t) \hat{\sigma}_+\right) \langle \hat{F}(\tau) \hat{F}^\dagger(0) \rangle + \left(\hat{\sigma}_- \hat{\sigma}_+ \hat{\rho}_A(t) - \hat{\sigma}_+ \hat{\rho}_A(t) \hat{\sigma}_-\right) \langle \hat{F}^\dagger(0) \hat{F}(\tau) \rangle + h.c \Big\}$$

Now, following the same algebra as in of the Wigner-Weisskopf approximation (Lecture 14 from Quantum Optics I).

$$\int_0^\infty d\tau \langle \hat{F}(\tau)\hat{F}^\dagger(0)\rangle = \int_0^\infty d\omega_k \overline{|g(\omega_k)|^2} \mathcal{D}(\omega_k)(\bar{n}(\omega_k)+1) \int_0^\infty d\tau\, e^{-i(\omega_k-\omega_{eg})\tau}$$

$$= \int_0^\infty d\omega_k \overline{|g(\omega_k)|^2} \mathcal{D}(\omega_k)(\bar{n}(\omega_k)+1)\, \zeta(\omega_k-\omega_{eg})$$

$$\zeta(\Delta) = \pi\delta(\Delta) - i\,\underbrace{\mathbb{P}\left(\frac{1}{\Delta}\right)}$$ Cauchy Principle Part

$$\Rightarrow \int_0^\infty d\tau \langle \hat{F}(\tau)\hat{F}^\dagger(0)\rangle = (\bar{n}(\omega_{eg})+1)\frac{\Gamma}{2} - \frac{i}{\hbar}(\delta E_{\bar{n}+1})$$

$$\Gamma = 2\pi\,\overline{|g(\omega_{eg})|^2}\,\mathcal{D}(\omega_{eg}) = \frac{4}{3}\frac{|\vec{d}_{eg}|^2}{\hbar}\left(\frac{\omega_{eg}}{c}\right)^3 = \text{Einstein A-coefficient}$$

$$\delta E_{\bar{n}+1} = \mathbb{P}\int_0^\infty \frac{[\bar{n}(\omega_k)+1]\,\overline{|g(\omega_k)|^2}}{\omega_{eg}-\omega_k}\,d\omega_k = \text{Light-shift (including Lamb)}$$

Similarly, $\int_0^\infty d\tau \langle \hat{F}^\dagger(0)\hat{F}(\tau)\rangle = \bar{n}(\omega_{eg})\frac{\Gamma}{2} + \frac{i}{\hbar}(\delta E_{\bar{n}})$

Note: If we look at the real parts of the correlation functions,

$$\mathrm{Re}\langle \hat{F}^\dagger(0)\hat{F}(\tau)\rangle = \Gamma\,\bar{n}(\omega_{eg})\,\delta(\tau)$$

$$\mathrm{Re}\langle \hat{F}(\tau)\hat{F}^\dagger(0)\rangle = \Gamma(\bar{n}(\omega_{eg})+1)\,\delta(\tau)$$

Note $\int_0^\infty d\tau\,\delta(\tau) = \frac{1}{2}$ ($\frac{1}{2}$ of a delta function when at boundary).

These have the characteristic "delta-correlations" associated with the Markov approximation. In addition, we see that the "strength" of the correlations, that is set by the "area" under the correlation curve, is proportional to the decay constant Γ. This is an important result, originally derived by Einstein in the context of classical Brownian motion, which relates the strength of the fluctuations in the reservoir to the decay rate. This is known as the Fluctuation-Dissipation theorem.

We can now plug this back into the equation of motion of ρ_S. We treat Δt now as a coarse-grained differential

$$\boxed{\begin{aligned}\frac{d\hat{\rho}_A}{dt} = -\frac{i}{\hbar}[H_A + \delta H_{LS}, \hat{\rho}_A] - \frac{\Gamma}{2}(\bar{n}+1)(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_A + \hat{\rho}_A\hat{\sigma}_+\hat{\sigma}_-) + \Gamma(\bar{n}+1)\hat{\sigma}_-\hat{\rho}_A\hat{\sigma}_+ \\ -\frac{\Gamma}{2}\bar{n}(\hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_A + \hat{\rho}_A\hat{\sigma}_-\hat{\sigma}_+) + \Gamma\bar{n}\,\hat{\sigma}_+\hat{\rho}_A\hat{\sigma}_-\end{aligned}}$$

This is the master equation for the atom in the presence of a thermal reservoir. When $\bar{n}\to 0$ we recover the standard form of a two level atom in the vacuum.

This equation is, as expected, in Lindblad form. There are two Lindblad jump operators

$\hat{L}_{abs} = \sqrt{\bar{n}\Gamma}\,\hat{\sigma}_+$, $\hat{L}_{emis} = \sqrt{(\bar{n}+1)\Gamma}\,\hat{\sigma}_-$, corresponding to absorption and emission of the atom from/to the field

$$\gamma_{e\leftarrow g} = |\langle e|\hat{L}_{abs}|g\rangle|^2 = \bar{n}\Gamma \quad (\text{"stimulated absorption"})$$

$$\gamma_{g\leftarrow e} = |\langle g|\hat{L}_{emis}|e\rangle|^2 = (\bar{n}+1)\Gamma \quad (\text{spontaneous + stimulated emission})$$

Including addition Hamiltonian dynamics

If we return to the Schrodinger picture our master equation for the atom

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_A, \hat{\rho}] + \mathcal{L}_{diss}[\hat{\rho}]$$

This represents the combination of the free evolution of the atom plus the dissipative effects to due spontaneous emission into the vacuum. But, what if there are additional forces driving the atom, e.g., a laser field tuned near an atomic resonance that drives coherent Rabi oscillations. Typically we just add in this additional interaction to the master eqn. So

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_A + \hat{H}_{AL}(t), \hat{\rho}\right] + \mathcal{L}_{diss}[\hat{\rho}]$$

In the rotating wave approximation, this then yields the optical bloch equations we studied last semester in Lectures #7 and #8.

When is the approach justified? Clearly we would require that the coupling of the system to the environment (here atom to vacuum fluctuations) are not affected by the additional interaction. This will be true when:

- the quantum fluctuations in the new interaction are equivalent to those in the enviroment.
- The dynamics induced by the new interaction are slow compared to the system-environment correlation time.

In the case of the atom-laser + vacuum interaction, these conditions hold to an excellent approximation. When the laser is a coherent state (or statistical mixture there of), the quantum fluctions are equivalent to that of the vacuum. In fact, as we have seen, we can make the Mollow transformation, and transform the interaction to look like the atom driven by a classical field, plus vacuum interaction. Moreover, the characteristic scale for dynamics of the atom is the Rabi frequency $\Omega = \vec{d}\cdot\vec{E}_L/\hbar$. Such frequency is typically of order of a few MHz, while the atom-vacuum correlation time is typically shorter than the optical period $T = \frac{2\pi}{\omega_L} \sim 100\,THz$.

Thus, we can add in the atom-laser interaction without having to re-derive the Master equation from scratch.

Note, however, when interacting with nonclassical light we must be more careful. The change in the nature of the quantum fluctuations can affect the nature of the dissipative terms in the master equation. Thus, for example, an atom is "Squeezed vacuum" will decay differently from an atom in the normal vacuum. In practice, though, in order to see such changes, the bandwidth of squeezed must be very large — this is rarely the case, and it is difficult to engineer such a reservoir.