Lecture 10: Master Equation: Examples

Last lecture we introduced the Lindblad form of the master equation, the most general Markov equation consistent with CP-maps:

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{L}_{relax}[\hat{\rho}]$$

$$\mathcal{L}_{relax} = \sum_{\mu}\left[-\frac{1}{2}\left(\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\hat{\rho} + \hat{\rho}\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\right) + \hat{L}_{\mu}\hat{\rho}\hat{L}_{\mu}^{\dagger}\right]$$

The set $\{\hat{L}_{\mu}\}$ are the Lindblad "jump operators" with $\gamma^{\mu}_{j\to j'} = |\langle j'|\hat{L}_{\mu}|j\rangle|^2$ the transition rate from $|j\rangle \to |j'\rangle$ according to a process μ.

Example 1: Two-level atom in a reservoir of black-body radiation

A canonical problem in quantum optics is a two level atom coupled to thermal reservoir of black-body radiation. Including the quantum fluctuations, this also include spontaneous emission in the vacuum (zero-temperature reservoir). The Lindblad equation follow from the usual system+environment Born-Markov approximation, with

$$\hat{H}_{total} = \underbrace{\frac{\hbar\omega_A}{2}\hat{\sigma}_z}_{H_S} + \underbrace{\sum_k \hbar\omega_k a_k^\dagger a_k}_{H_E} + \underbrace{\hbar\sum_k \left(g_k a_k \sigma_+ + g_k^* a_k^\dagger \sigma_-\right)}_{H_{SE}}$$

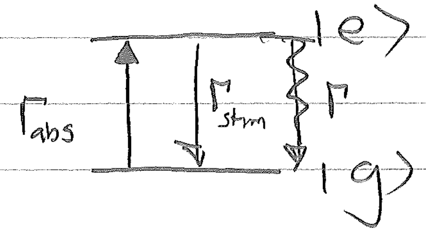
The field is the "environment" in a thermal state

$$\hat{\rho}_E(0) = \prod_k \frac{e^{-\beta\hbar\omega_k a_k^\dagger a_k}}{Z_k} = \prod_k \frac{\bar{n}_k^{\,n_k}}{(\bar{n}_k+1)^{n_k+1}} |n_k\rangle\langle n_k|$$

where $\beta = \frac{1}{k_B T}$, $Z_k = \frac{1}{1-e^{-\beta\hbar\omega_k}}$, $\bar{n}_k = \frac{1}{e^{\beta\hbar\omega_k}-1}$

Note: at $\beta \to \infty$ $(T=0)$ $\hat{\rho}_E(0) \to |vac\rangle\langle vac|$

The interaction of the atom and field leads to absorption + emission



Γ_{abs} = absorption rate

Γ_{stim} = stimulated emission rate

Γ = spontaneous emission rate

According to the Einstein A-B relations

$$\Gamma_{abs} = \Gamma_{stim} = \bar{n}\Gamma$$

$$\text{where } \bar{n} = \bar{n}(\omega_{eg}) = \frac{1}{e^{\beta\hbar\omega_{eg}}-1}$$

We thus have two Lindblad operators defined by

$$\Gamma_{abs} = |\langle e|\hat{L}_{abs}|g\rangle|^2 = \bar{n}\Gamma \Rightarrow \hat{L}_{abs} = \sqrt{\bar{n}\Gamma}\,\hat{\sigma}_+$$

$$\Gamma_{ems} = |\langle g|\hat{L}_{emiss}|e\rangle|^2 = (\bar{n}+1)\Gamma \Rightarrow \hat{L}_{emis} = \sqrt{(\bar{n}+1)\Gamma}\,\hat{\sigma}_-$$

We thus have the Master Eqn for the Atom

$$\frac{d\hat{\rho}_A}{dt} = -\frac{i}{\hbar}[\hat{H}_A, \hat{\rho}_A] + \mathcal{L}_{relax}[\hat{\rho}_A]$$

$$\mathcal{L}_{relax}[\hat{\rho}_A] = -\frac{\Gamma}{2}(\bar{n}+1)\left(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_A + \hat{\rho}_A\hat{\sigma}_+\hat{\sigma}_- - 2\hat{\sigma}_-\hat{\rho}_A\hat{\sigma}_+\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_A + \hat{\rho}_A\hat{\sigma}_+\hat{\sigma}_- - 2\hat{\sigma}_+\hat{\rho}_A\hat{\sigma}_-\right)$$

$$\hat{H}_A = \hbar\omega_{eg}|e\rangle\langle e| = \hbar\omega_{eg}\hat{\sigma}_+\hat{\sigma}_-$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -i\omega_{eg}[|e\rangle\langle e|, \hat{\rho}_A] - \frac{\Gamma}{2}(\bar{n}+1)\left(\{|e\rangle\langle e|, \hat{\rho}\} - 2|g\rangle\langle g|\rho_{ee}\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\underset{\uparrow \text{ anti-commutator}}{\{|g\rangle\langle g|, \hat{\rho}\}} - 2|e\rangle\langle e|\rho_{gg}\right)$$

Evolution of matrix elements

$$\frac{d}{dt}\rho_{ee} = \frac{d}{dt}\langle e|\hat{\rho}|e\rangle = \underbrace{-\Gamma(\bar{n}+1)}_{\text{emission}}\rho_{ee} + \underbrace{\Gamma\bar{n}}_{\text{absorbtion}}\rho_{gg}$$

$$\frac{d}{dt}\rho_{gg} = \frac{d}{dt}\langle g|\hat{\rho}|g\rangle = -\Gamma\bar{n}\rho_{gg} + \Gamma(\bar{n}+1)\rho_{ee}$$

Trace preserving $\frac{d}{dt}(\rho_{gg} + \rho_{ee}) = 0$

Steady State $\Rightarrow$ detailed balance

$$\Rightarrow \frac{d}{dt}\hat{\rho} = 0 \Rightarrow \frac{\rho_{ee}}{\rho_{gg}} = \frac{\bar{n}}{\bar{n}+1} = \frac{(e^{\beta\hbar\omega_{eg}}-1)^{-1}}{(e^{\beta\hbar\omega_{eg}}-1)^{-1}+1}$$

$$= e^{-\beta\hbar\omega_{eg}}$$ Boltzmann! ✓

Thus, in steady state the atom come to equilibrium with the bath, as expected

In fact, Einstein <u>derived</u> the spontaneous emission rate to get thermal equilibrium (see: Einstein A/B coefficients)

Decay of coherences (in the absence of coherent driving)

$$\frac{d}{dt}\rho_{eg} = \frac{d}{dt}\langle e|\hat{\rho}|g\rangle$$

$$= -i\omega_{eg}\,\rho_{eg} - \frac{\Gamma}{2}(2\bar{n}+1)\,\rho_{eg}$$

Decay of coherences $\gamma = \dfrac{\Gamma_e + \Gamma_g}{2}$

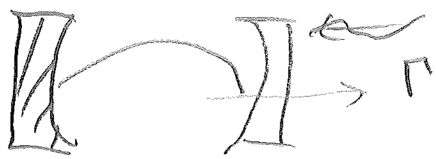
(in the absence of collisions)

Note: Without coherent drives <u>separate</u> equations of coherences and populations

Another example: Damped SHO

Given oscillator @ freq ω_0 coupled to a bath of thermal oscillators

E.g. mode in leaky cavity



$\hat{L}_{abs} = \sqrt{\Gamma \bar{n}}\, \hat{a}^\dagger$

$\hat{L}_{emis} = \sqrt{\Gamma(\bar{n}+1)}\, \hat{a}$

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}(\bar{n}+1)\left[\{\hat{a}^\dagger\hat{a},\hat{\rho}\} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right]$$
$$-\frac{\Gamma}{2}\bar{n}\left[\{\hat{a}\hat{a}^\dagger,\hat{\rho}\} - 2\hat{a}^\dagger\hat{\rho}\hat{a}\right]$$

Derived in same Born-Markov approx with

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger\hat{a} + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k$$
$$+ \sum_k \left(g_k \hat{b}_k^\dagger \hat{a} + g_k^* \hat{b}_k \hat{a}^\dagger\right)$$

linear coupling

In condensed matter literature known as "Caldera-Legget" model)

Here $\hat{H}_{int} = \hbar \hat{F}(t)\, \hat{a}^{\dagger} + \hbar \hat{F}^{\dagger}(t)\, \hat{a}$

Bath fluctuation $\hat{F}(t) = \sum_k g_k b_k e^{-i(\omega_k - \omega_0)t}$

$$Re\left(\langle \hat{F}(t_1)\, \hat{F}^{\dagger}(t_2)\rangle\right) = \frac{\Gamma}{2}(\bar{n}+1)\, \delta(t_1 - t_2)$$

$$Re\left(\langle \hat{F}^{\dagger}(t_1)\, \hat{F}(t_2)\rangle\right) = \frac{\Gamma}{2}\bar{n}\, \delta(t_1 - t_2)$$

Population rate equations $P_n \equiv \langle n|\hat{\rho}|n\rangle$

$$\frac{dP_n}{dt} = -\frac{\Gamma}{2}(\bar{n}+1)\left(\langle n|\hat{a}^{\dagger}\hat{a}\rho|n\rangle + \langle n|\hat{\rho}\hat{a}^{\dagger}\hat{a}|n\rangle\right) + \Gamma(\bar{n}+1)\langle n|\hat{a}\hat{\rho}\hat{a}^{\dagger}|n\rangle$$

$$+\frac{\Gamma}{2}\bar{n}\left(\langle n|\hat{a}\hat{a}^{\dagger}\hat{\rho}|n\rangle + \langle n|\hat{\rho}\hat{a}\hat{a}^{\dagger}|n\rangle\right) + \Gamma\bar{n}\langle n|\hat{a}\hat{\rho}\hat{a}^{\dagger}|n\rangle$$

$$= -\Gamma(\bar{n}+1)n P_n + \Gamma(\bar{n}+1)(n+1)P_{n+1}$$

$$-\Gamma\bar{n}(n+1)P_n + \Gamma\bar{n}n P_{n-1}$$

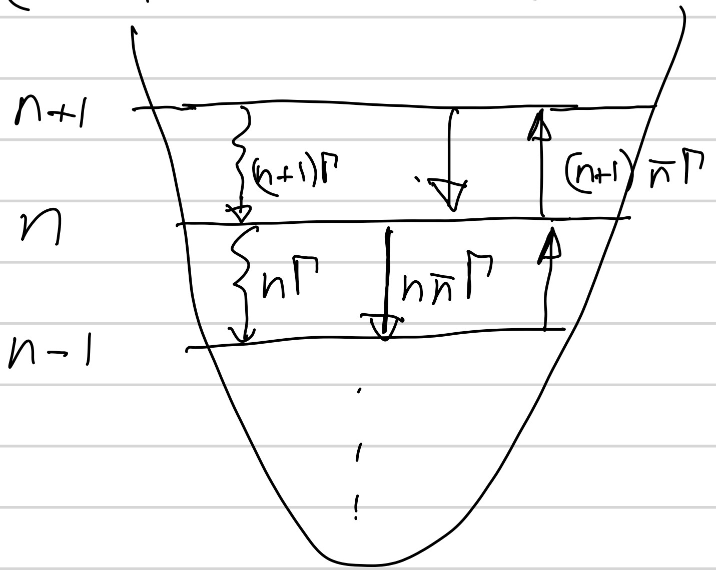
Populations couple only to populations

$$\gamma_{n+1\leftarrow n} = \sum_{\mu}|\langle n+1|\hat{L}_{\mu}|n\rangle|^2 = \Gamma\bar{n}|\langle n+1|\hat{a}^{\dagger}|n\rangle|^2 = \Gamma\bar{n}(n+1)$$

$$\gamma_{n-1\leftarrow n} = \Gamma(\bar{n}+1)|\langle n-1|\hat{a}|n\rangle|^2 = \Gamma(\bar{n}+1)n = n\Gamma + n(\bar{n}\Gamma)$$

$$\gamma_{n\leftarrow n-1} = \Gamma\bar{n}|\langle n|\hat{a}^{\dagger}|n-1\rangle|^2 = \Gamma\bar{n}n$$

$$\gamma_{n\leftarrow n+1} = \Gamma(\bar{n}+1)|\langle n|\hat{a}|n+1\rangle|^2 = \Gamma(\bar{n}+1)(n+1) = (n+1)\Gamma + (n+1)(\bar{n}\Gamma)$$



Consider the evolution of expectation values of observables:

Aside: $\frac{d}{dt}\langle\hat{A}\rangle = \frac{d}{dt}\mathrm{Tr}(\rho\hat{A}) = \mathrm{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{A}\right)$

$$\Rightarrow \frac{d}{dt}\langle\hat{A}\rangle = -\frac{1}{2}\sum_{\mu}\mathrm{Tr}\left(\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\hat{\rho}\hat{A} + \hat{\rho}\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\hat{A} - 2\hat{L}_{\mu}\rho\hat{L}_{\mu}^{\dagger}\hat{A}\right)$$

$$= -\frac{1}{2}\sum_{\mu}\mathrm{Tr}\left[\left(\hat{A}\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu} + \hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\hat{A} - 2\hat{L}_{\mu}^{\dagger}\hat{A}\hat{L}_{\mu}\right)\hat{\rho}\right]$$

$$\boxed{\frac{d}{dt}\langle\hat{A}\rangle = -\frac{1}{2}\sum_{\mu}\left(\langle\hat{L}_{\mu}^{\dagger}[\hat{L}_{\mu},\hat{A}]\rangle + \langle[\hat{A},\hat{L}_{\mu}^{\dagger}]\hat{L}_{\mu}\rangle\right)}$$

For example, in damped SHO: mean excitation

$$\frac{d}{dt}\langle\hat{n}\rangle = -\frac{\Gamma}{2}(\bar{n}+1)\left(\langle\hat{a}^{\dagger}[\hat{a},\hat{n}]\rangle + \langle[\hat{n},\hat{a}^{\dagger}]\hat{a}\rangle\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\langle\hat{a}[\hat{a}^{\dagger},\hat{n}]\rangle + \langle[\hat{n},\hat{a}]\hat{a}^{\dagger}\rangle\right)$$

$$= -\frac{\Gamma}{2}(\bar{n}+1)\left(+\langle\hat{a}^{\dagger}a\rangle + \langle\hat{a}^{\dagger}a\rangle\right) - \frac{\Gamma}{2}\bar{n}\left(-\langle\hat{a}\hat{a}^{\dagger}\rangle - \langle\hat{a}\hat{a}^{\dagger}\rangle\right)$$

$$= -\Gamma(\bar{n}+1)\langle\hat{n}\rangle + \Gamma\bar{n}\left(\langle\bar{n}\rangle + 1\right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{n}\rangle = -\Gamma\langle\hat{n}\rangle + \Gamma\bar{n}}$$

Solution: $\langle\hat{n}\rangle(t) = \langle\hat{n}\rangle(0) + \bar{n}\left(1-e^{-\Gamma t}\right)$

Steady state: $\boxed{\langle\hat{n}\rangle = \bar{n}\ :\ \text{Thermal equilibrium}}$

Coherences:

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{i}{\hbar}\underbrace{\langle[\hat{a},\hat{H}]\rangle}_{\hbar\omega_0\hat{a}} + \mathrm{Tr}\left(\mathcal{L}_{relax}[\hat{\rho}]\,\hat{a}\right)$$

$$\mathrm{Tr}\left(\mathcal{L}_{relax}[\hat{\rho}]\hat{a}\right) = -\frac{\Gamma}{2}(\bar{n}+1)\left(\langle\hat{a}^\dagger[\hat{a},\hat{a}]\rangle + \langle[\hat{a},\hat{a}^\dagger]\hat{a}\rangle\right)$$

$$-\frac{\Gamma}{2}\bar{n}\left(\langle\hat{a}[\hat{a}^\dagger,\hat{a}]\rangle + \langle[\hat{a},\hat{a}]\hat{a}^\dagger\rangle\right)$$

$$= -\frac{\Gamma}{2}(\bar{n}+1)\langle\hat{a}\rangle + \frac{\Gamma}{2}\bar{n}\langle\hat{a}\rangle = -\frac{\Gamma}{2}\langle\hat{a}\rangle$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{a}\rangle = \left(-i\omega_0 - \frac{\Gamma}{2}\right)\langle\hat{a}\rangle}$$ Decay amplitude

$$\Rightarrow \boxed{\langle\hat{a}\rangle(t) = \langle\hat{a}\rangle(0)\, e^{-i\omega_0 t - \frac{\Gamma}{2}t}}$$

<u>Note</u>: The rate of decay is independent of $\langle\hat{a}\rangle$. How is this possible since the decay of ρ_{nn} depends on n?

<u>Look at evolution of coherences of density op</u>

$$\frac{d}{dt}\rho_{n+1,n} = \left(-i\omega_0 - \left[2\bar{n} + \tfrac{1}{2} + \underset{\text{dependence on } n}{n}(2\bar{n}+1)\right]\Gamma\right)\rho_{n+1,n}$$

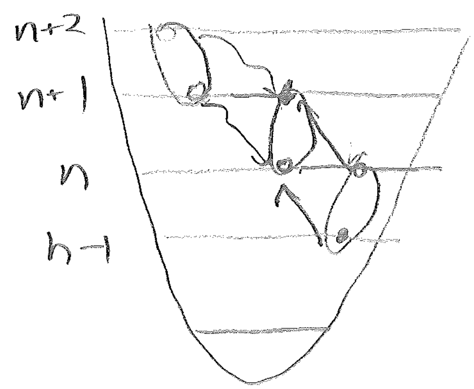
$$+\sqrt{(n+1)(n+2)}\,(\bar{n}+1)\Gamma\,\rho_{n+2,n+1}$$

$$+\sqrt{n(n+1)}\,\bar{n}\,\rho_{n,n-1}$$

Note: Coherence decay at rate depending on n

BUT there are also feeding terms

Transfer of coherence



Coherent superposition of $|n+2\rangle$ and $|n+1\rangle$ transfered to superposition of $|n+1\rangle$ and $|n\rangle$

This is only possible because the two decay paths are indistinguishable

This is true only for harmonic ladder

Where the spacing between levels is equal.

To see how the transfer of coherences removes the dependence of the decay of the mean field on n, consider the case of zero temperature $\bar{n} = 0$

$$\Rightarrow \frac{d}{dt} \rho_{n+1,n} = -(n+\tfrac{1}{2})\Gamma \rho_{n+1,n} + \underbrace{\sqrt{(n+2)(n+1)}\,\Gamma \rho_{n+2,n+1}}_{\text{Feeding of coherences}}$$

Mean field $\langle \hat{a} \rangle(t) = Tr(\hat{a}\hat{\rho}(t)) = \sum_n \langle n | \hat{a}\, \hat{\rho}(t) | n \rangle$

$$= \sum_n \sqrt{n+1}\, \rho^{(t)}_{n+1,n}$$

$$\Rightarrow \frac{d}{dt} \langle \hat{a} \rangle(t) = \sum_{n=0}^{\infty} \sqrt{n+1}\, \frac{d\rho_{n+1,n}}{dt}$$

$$= \underbrace{-\Gamma \sum_{n=0}^{\infty} \sqrt{n+1}\,(n+\tfrac{1}{2})\,\rho_{n+1,n}}_{-\frac{\Gamma}{2}\langle a \rangle + \sum_{n=1}^{\infty} \sqrt{n+1}\, n\, \rho_{n+1,n}} + \underbrace{\Gamma \sum_{n=0}^{\infty} \sqrt{n+2}\,(n+1)\,\rho_{n+2,n+1}}_{+\Gamma \sum_{n=1}^{\infty} \sqrt{n+1}\, n\, \rho_{n+1,n}}$$

$$\Rightarrow \frac{d}{dt} \langle \hat{a} \rangle(t) = -\frac{\Gamma}{2} \langle \hat{a} \rangle$$

$\Rightarrow$ For a damped SHO, the mean field decays at a rate independent of $\langle a \rangle$

Evolution of quadratures

$$\hat{X}_\phi = \frac{\hat{a} e^{-i\phi} + \hat{a}^\dagger e^{i\phi}}{\sqrt{2}}$$

$$\Rightarrow \frac{d}{dt}\langle \hat{X}_\phi \rangle = -\frac{\Gamma}{2}\langle \hat{X}_\phi \rangle \quad \text{(in rotating frame)}$$

$$\frac{d}{dt}\langle \hat{X}_\phi^2 \rangle = -\frac{\Gamma}{2}(\bar{n}+1)\left(\langle \hat{a}^\dagger [\hat{a}, \hat{X}_\phi^2]\rangle + \langle [\hat{X}_\phi^2, \hat{a}^\dagger]\, \hat{a}\rangle\right)$$
$$-\frac{\Gamma}{2}\bar{n}\left(\langle \hat{a}[\hat{a}^\dagger, \hat{X}_\phi^2]\rangle + \langle [\hat{X}_\phi^2, \hat{a}]\, \hat{a}^\dagger\rangle\right)$$

Aside: $[\hat{a}, \hat{X}_\phi^2] = \hat{X}_\phi [\hat{a}, \hat{X}_\phi] + [\hat{a}, \hat{X}_\phi]\hat{X}_\phi$
$= \sqrt{2}\hat{X}_\phi e^{i\phi}$

$$\Rightarrow \frac{d}{dt}\langle \hat{X}_\phi^2 \rangle = -\frac{\Gamma}{\sqrt{2}}(\bar{n}+1)\langle \hat{a}^\dagger e^{i\phi}\hat{X}_\phi + \hat{X}_\phi \hat{a} e^{-i\phi}\rangle$$
$$-\frac{\Gamma}{\sqrt{2}}\bar{n}\langle -\hat{a} e^{-i\phi}\hat{X}_\phi - \hat{X}_\phi \hat{a}^\dagger e^{i\phi}\rangle$$
$$= -\frac{\Gamma}{\sqrt{2}}(\bar{n}+1)\left[\sqrt{2}\langle \hat{X}_\phi^2\rangle - \frac{1}{\sqrt{2}}\right]$$
$$+\frac{\Gamma}{\sqrt{2}}\bar{n}\left[\sqrt{2}\langle \hat{X}_\phi^2\rangle + \frac{1}{\sqrt{2}}\right]$$
$$= -\Gamma\langle \hat{X}_\phi^2\rangle + \frac{\Gamma}{2}(2\bar{n}+1)$$

$$\Rightarrow \frac{d}{dt}\langle \Delta\hat{X}_\phi^2\rangle = \frac{d}{dt}\left(\langle \hat{X}_\phi^2\rangle - \langle \hat{X}_\phi\rangle^2\right)$$
$$= -\Gamma\langle \Delta\hat{X}_\phi^2\rangle + \frac{\Gamma}{2}(2\bar{n}+1)$$

Solution:

$$\boxed{\langle \Delta \hat{X}_\phi^2 \rangle (t) = \langle \Delta \hat{X}_\phi^2 \rangle (0)\, e^{-\Gamma t} + \left(1 - e^{-\Gamma t}\right)\left(\frac{2\bar{n}+1}{\cancel{4}2}\right)}$$

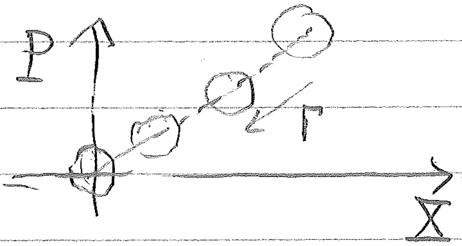
In steady state $\langle \Delta \hat{X}_\phi^2 \rangle = \frac{1}{\cancel{4}2}(2\bar{n}+1)$

Note: Even at zero temperature, the vacuum with $\bar{n}=0$, the quadrature fluctuations damp

$$\langle \Delta \hat{X}_\phi^2 \rangle (t) = \langle \Delta \hat{X}_\phi^2 (0) \rangle e^{-\Gamma t} + \frac{1}{2\cancel{4}}\left(1 - e^{-\Gamma t/2}\right)$$

- Example: Coherent State $\langle \Delta \hat{X}_\phi^2 \rangle (0) = \frac{1}{\cancel{4}2}$

$$\Rightarrow \langle \Delta \hat{X}_\phi^2 \rangle (t) = \frac{1}{\cancel{4}2} \quad \forall t$$

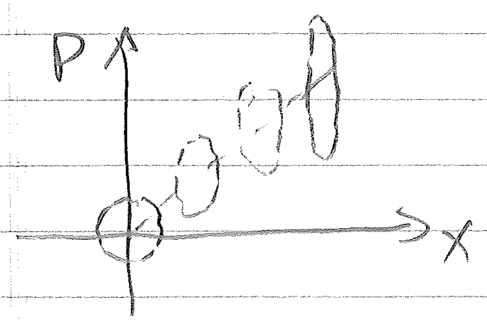


For a coherent state mean amplitude damps but fluctuations unchange

Coherent state is eigenstate of Linblad operator
$\hat{L} = \sqrt{\Gamma}\,\hat{a} \Rightarrow$ "pointer state"

- Example Squeezed State $\langle \Delta \hat{X}^2 (0) \rangle = \frac{e^{-2r}}{\cancel{4}2}$

$$\langle \Delta \hat{P}^2 (0) \rangle = \frac{e^{+2r}}{2\cancel{4}}$$



$\boxed{\text{Damping kill SQUEEZING}}$

Squeezed state depend on correlated photons.