Physics 581: Open Quantum Systems

Lecture 13: Stochastic Methods and the Quantum/Classical Boundary

We have seen that for a classical (Markovian) open quantum system, we have two pictures,

(1) Liouville Picture: Probability density evolving as a function of time in phase space: $P(\vec{q},\vec{p},t)$

$$\frac{\partial}{\partial t} P(\vec{q},\vec{p},t) = \{H, P\} + \mathcal{L}[P]$$

(2) Hamilton-Langevin Picture: Trajectories $(\vec{q}(t), \vec{p}(t))$ evolving stochastically according to a random Force

$$\frac{d\vec{q}}{dt} = \{\vec{q}, H\} \quad , \quad \frac{d\vec{p}}{dt} = \{\vec{p}, H\} + \mathcal{L}^{\dagger}[\vec{p}]$$

e.g. Brownian motion with damping: $\frac{d\vec{p}}{dt} = -\gamma\vec{p} + \vec{\mathcal{F}}(t)$ Gaussian white noise $\langle \mathcal{F}_i(t_1)\, \mathcal{F}_j(t_2)\rangle = D_{ij}\, \delta(t_1 - t_2)$ Diffusion matrix

In either case we can calculate average values

$$\langle A(\vec{q},\vec{p})\rangle(t) = \int d\vec{q}\, d\vec{p}\; P(\vec{q},\vec{p},t)\, A(\vec{q},\vec{p}) = \overline{A(\vec{q}(t), \vec{p}(t))}$$

$\leftarrow$ Average over statistics of random force

We have also seen, in the context of CV quantum mechanics, we can express the master equation in a phase space representation, e.g., for the Wigner function

$$\frac{\partial}{\partial t} W(\vec{q},\vec{p},t) = \{H, W\}_{MB} + \mathcal{L}[W]$$

A question is, under what circumstances can we approximately solve for $W(\vec{q},\vec{p},t)$ by simulating a set of classically stochastic trajectories? Here the noisy force can arise both from classical thermal fluctuations and quantum fluctuations in the environment. This is potential a useful (approximate) calculational tool as well as a useful formalism for understanding the quantum/classical boundary.

To study this let us first review some foundations of classical stochastic processes

Stochastic Processes

A stochastic process describes a random variable as a function of time $X(t)$
assign a probability (or density if X is a continuous variable) $p(x,t)$ = probability $x = X(t)$ at time t
The joint probability $p(x_1,t_1; x_2,t_2; \cdots, x_N,t_N)$ = Probability that $X(t_1)=x_1$ and $X(t_2)=x_2$ and $\cdots$ $X(t_N)=x_N$

Markov process

A stochastic process in Markovian if

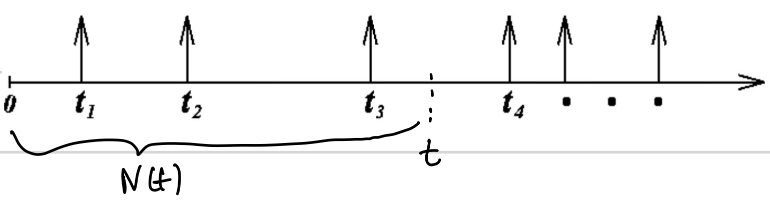
$$p(x_N, t_N \mid x_{N-1}, t_{N-1}, \cdots x_2 t_2, x_1 t_1) = p(x_N, t_N \mid x_{N-1}, t_{N-1}) \qquad t_N > t_{N-1} \cdots > t_1$$

$$\Rightarrow p(x_1, t_1, \cdots x_N, t_N) = p(x_N, t_N \mid x_{N-1}, t_{N-1})\, p(x_{N-1}, t_{N-1} \mid x_{N-2}, t_{N-2}) \cdots p(x_2 t_2 \mid x_1, t_1)\, p(x_1, t_1)$$

Examples of Markov processes

- Poisson Process

A Poisson process describes the counting of random events that arrive at the counter at a constant rate, but event event is random and completely uncorrelated from any other event



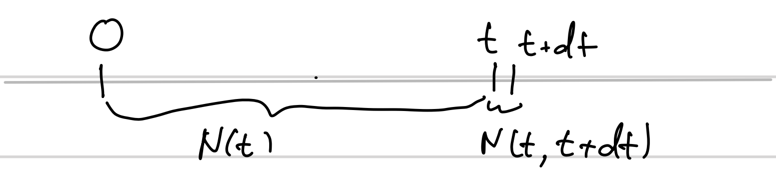
(Most randomly spaced events)

This is a stationary, ergodic, Markov process.

We define the random varible $N(t) = \#$ of counts $[0, t)$ and $N(t_a, t_b) = \#$ between t_a and t_b $= N(t_a) - N(t_b)$. $N(t)$ is a process with rate λ in two equivalent definitions:

(1) The Probability distribution of $N(t_a, t_b)$ is a Poisson distribution with mean $\lambda(t_b - t_a)$, and every nonoverlapping interval is statistically independent.

(2) In an infinitesimal time interval dt there can be no more than one count with probality $P(N(t, t+dt) = 1) = \lambda dt$, statically independent of the arrivals outside of this interval.



$$\Rightarrow p(n, t+dt \mid n-1, t) = \lambda dt \quad \Rightarrow \quad P(n, t+dt \mid 0, 0) = \underbrace{P(n, t+dt \mid n-1, t)}_{\lambda dt} P(n-1, t \mid 00) + \overbrace{P(n, t+dt \mid n, t)}^{1-\lambda dt} * P(n, t \mid 00)$$

$$p(n, t+dt \mid n, t) = 1 - \lambda dt$$

$$\boxed{\frac{d}{dt} P(n,t|0,0) = \lambda P(n-1,t|0,0) - \lambda P(n,t|0,0)}$$ Poisson process

Solution: $P(n,t|0,0) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ Poisson distribution

$$\langle n\rangle(t) = \sum_{n=0}^{\infty} n\, P(n,t|0,0) = \lambda t$$

$$\langle \Delta n^2\rangle(t) = \sum_{n=0}^{\infty} (n-\langle n\rangle)^2 P(n,t|0,0) = \lambda t \qquad \sqrt{\Delta n^2(t)} = \sqrt{\lambda t}$$

Weiner Process

In a Weiner process the stochastic variable $X(t) \in \mathbb{R}$ (continuous variable). We can think of this as the continuum limit of discrete random walk. In the classic problem the walker moves to the left or the right randomly in time interval τ (say 50-50 probability). We seek the probability of makin n steps to the right after time t $p(n,t|0,0)$

$$p(n,(N+1)\tau|n-1,N\tau) = p(n,(N+1)\tau|n+1,N\tau) = \frac{1}{2}$$

$$\underbrace{p(n,(N+1)\tau|n',N'\tau)}_{\tau\to\Delta t,\ t=N\tau,\ t'=N'\tau} = \underbrace{P(n,(N+1)\tau|n-1,N\tau)}_{1/2} P(n-1\ N\tau|n',N'\tau) + \underbrace{P(n,(N+1)\tau|n+1,N\tau)}_{\frac{1}{2}} P(n+1,N\tau|n',N'\tau)$$

$$\approx p(n,t|n',t') + \frac{\partial}{\partial t}P(n,t|n',t')\tau = \frac{1}{2}P(n-1,t|n',t') + \frac{1}{2}P(n+1,t|n',t')$$

Random walk $$\boxed{\frac{\partial}{\partial t}P(n,t|n',t) = \frac{\Lambda}{2}P(n-1,t|n',t') + \frac{\Lambda}{2}P(n+1,t|n',t) - \Lambda P(n,t|n',t) \qquad \Lambda = \frac{1}{\Delta\tau} = \text{rate of transitions}}$$

Let $W = n\Delta W$. In the limit $\Delta W \to 0$ this finite difference equation become a partial differential equation

$$\frac{\partial}{\partial t}P(w,t|0,0) = \frac{D}{2}\frac{\partial^2 P}{\partial W^2} \quad \text{where} \quad D = \lim_{\tau\to 0}\lim_{\Delta x\to 0}\frac{\Delta W^2}{\Delta t} \text{ (Finite)} = \text{Diffusion coefficient}$$

Solution: $P(w,t|0,0) = \frac{1}{\sqrt{2\pi Dt}} e^{-\frac{W^2}{2Dt}}$, more generally $P(w,t|w_0,t_0) = \frac{1}{\sqrt{2\pi D(t-t_0)}} e^{-\frac{(w-w_0)^2}{2D(t-t_0)}}$

Gaussian, mean w_0, Variance $D(t-t_0)$

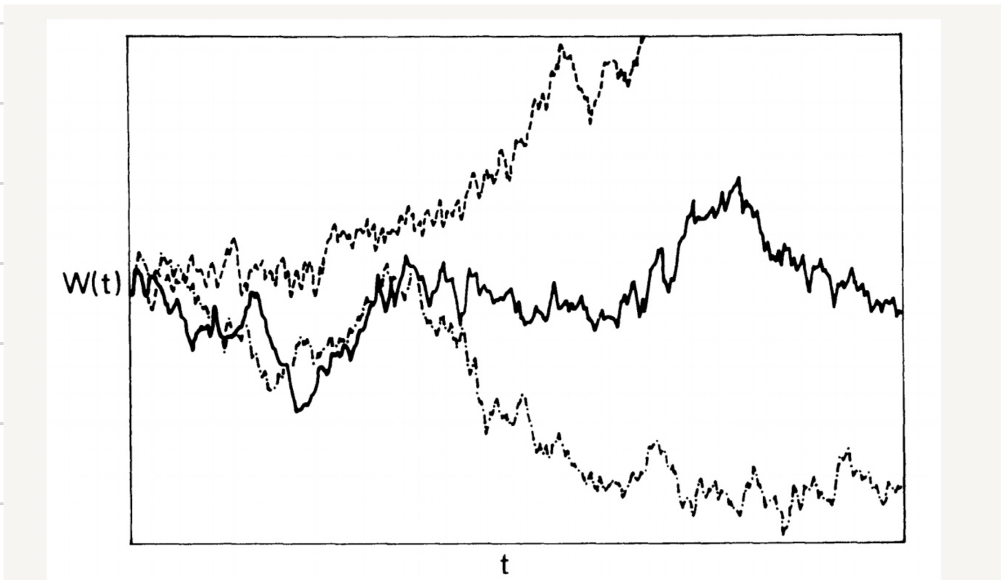
The Weiner Process is defined such that $D = \lim_{\substack{\Delta W\to 0\\ \Delta t\to 0}} \frac{\Delta W^2}{\Delta t} = 1$ $\quad (dW)^2 = dt$

$$\boxed{W(t) \text{ distributed by } p(w,t|w_0,t_0), \text{ satisfying } \frac{\partial}{\partial t}p(w,t|w_0,t_0) = \frac{1}{2}\frac{\partial^2 p}{\partial w^2}}$$

$$p(w,t|w_0,t_0) = \frac{1}{\sqrt{2\pi(t-t_0)}} e^{-\frac{(w-w_0)^2}{2(t-t_0)}} \qquad \langle W(t)\rangle = W_0 \qquad \langle \Delta W^2\rangle(t) = t-t_0$$

typical take $t_0 = 0$, $w_0 = 0$

The Weiner process is an example of a Markov process with continuous sample paths. But is has some surprising properties. The sample paths are highly irregular. Although the mean is zero the Variance $\to \infty$ at $t \to \infty$.



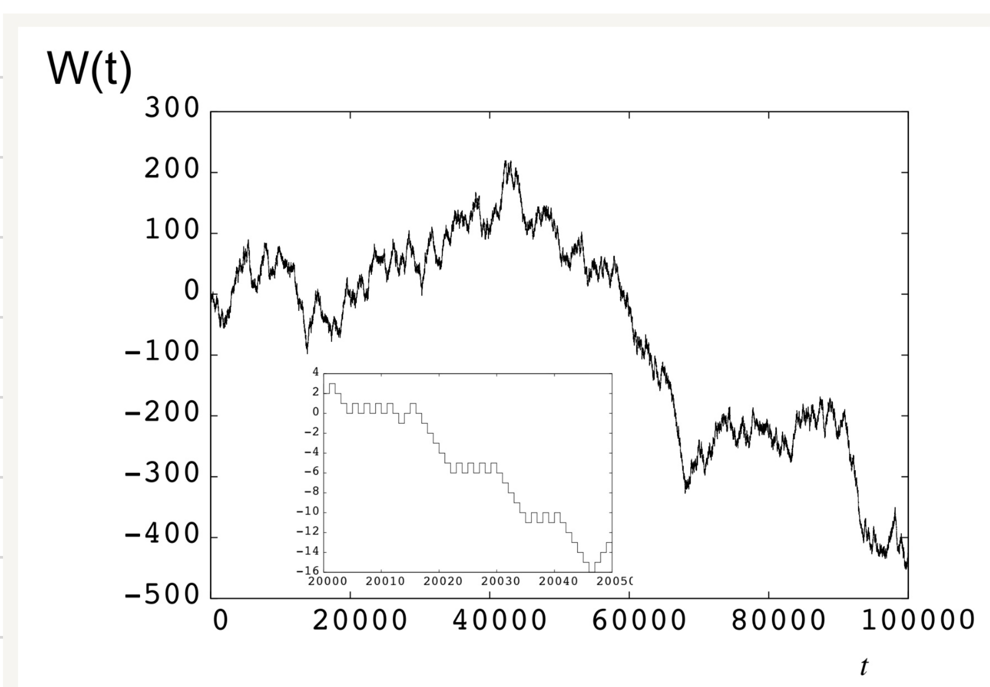
Three simulated sample Paths, $W(t)$ showing their great variability (from Gardiner)

While the path are continuous they are not differentiable. Consider

$$\text{Prob}\left[\, |W(t+\Delta t) - W(t)| > k\Delta t \right] = 2\int_{k\Delta t}^{\infty} dw \, \frac{e^{-\frac{w^2}{2h}}}{\sqrt{2\pi h}} \to 1 \quad \text{as} \quad \Delta t \to 0$$

$\Rightarrow$ $\left|\dfrac{W(t+\Delta t) - W(t)}{\Delta t}\right|$ is almost certainly larger than k, no matter what k we chose
(remember, $dw \sim \sqrt{dt}$)

Physically, their are no real continuous markov processes. We obtained this here through a limit of discrete random walks, with discontinuous jumps.



The Orstein – Uhlenbeck Process describes the drag and diffusion of Brownian motion we have consider from point of view of the Langenvin equation. In this case the probability distribution evolves according to a Fokker – Planck equation

$$\boxed{\frac{\partial}{\partial t} p(x,t|x_0,t_0) = \frac{\partial}{\partial x}\left(\frac{\Gamma}{2} x p\right) + \frac{D}{2}\frac{\partial^2 p}{\partial x^2}, \qquad p(x,t_0|x_0 t_0) = \delta(x-x_0)}$$

With solution $p(x,t|x_0,t_0) = \frac{1}{\sqrt{2\pi \Delta x^2(t)}} e^{-\frac{(x-\bar{x}(t))^2}{2\Delta x^2(t)}}$

$$\langle x(t)\rangle = x_0 e^{-\frac{\Gamma}{2}t} \qquad \langle \Delta x^2(t)\rangle = \frac{D}{\Gamma}\left(1-e^{-\Gamma t}\right) \rightarrow \begin{cases} Dt & \text{short time} \\ \frac{D}{\Gamma} & \text{asymptotically} \end{cases}$$

This process also has continuous (non differentiable) paths in $x(t)$

For the stochastic process given here, we have boxed these equation of motion are known in classical statistical physics as the "master equation," which is where the quantum analog for ρ gets its name. We see here rate equations for the discrete random variables as PDE's where the stochastic variable was a continuous function of time. We seek now to understand the general form such equations can take for Markov processes.

Chapmann – Kolmogorov equation

Consider a Markov process. $p(x,t|x_0,t_0) = \int dx_1\, p(x,t; x_1 t_1 | x_0 t_0)$ marginalization

$$= \int dx_1\, p(x,t|x_1,t_1; x_0 t_0)\, p(x_1,t_1|x_0,t_0) \quad \text{Conditional prob}$$

$$\boxed{\text{Integral form:} \quad p(x,t|x_0 t_0) = \int dx_1\, p(x,t|x_1,t_1)\, p(x_1,t_1|x_0 t_0) \qquad t_1 > t_0} \quad \text{Markov}$$

Let $\Lambda_t(x|x') = \lim_{\Delta t\to 0} \frac{p(x,t+\Delta t|x',t)}{\Delta t}$ transition rates

$$\boxed{\text{Differential form:} \quad \frac{\partial}{\partial t} p(x,t|x_0 t_0) = \int dx' \Big(\underbrace{\Lambda_t(x|x')}_{\text{rate of transition } x \leftarrow x'} p(x',t|x_0 t_0) - \underbrace{\Lambda_t(x'|x)}_{\text{rate of transition } x' \leftarrow x} p(x,t|x_0,t_0)\Big)}$$

Master equation

Note: We can also write this for the unconditioned prob $p(x,t) = \int dx_0\, p(x,t; x_0 t_0)$

$$= \int dx_0\, p(x,t|x_0 t_0)\, p(x_0,t_0)$$

$\Rightarrow \frac{\partial}{\partial t} p(x,t) = \int dx' \left(\Lambda_t(x|x') p(x',t) - \Lambda_t(x'|x) p(x,t)\right)$ with initial condition $p(x,t_0)$

Now let $x' = x - r$, $r = x - x'$

$$\Rightarrow \frac{\partial}{\partial t} p(x,t) = \int_{-\infty}^{\infty} dr \left[\Lambda_t(x|x-r)\, p(x-r,t) - \Lambda_t(x-r|x)\, p(x,t) \right]$$

Let $\tilde{\Lambda}_t(x,r) = \Lambda_t(x+r|x)$

$$\Rightarrow \frac{\partial}{\partial t} p = \int_{-\infty}^{\infty} dr \left[\tilde{\Lambda}_t(x-r,r)\, p(x-r,t) - \tilde{\Lambda}_t(x,-r)\, p(x,t) \right] = \int_{-\infty}^{\infty} dr \left[\underbrace{\tilde{\Lambda}_t(x-r,r)\, p(x-r,t)}_{f(x-r,t)} - \underbrace{\tilde{\Lambda}_t(x,r)\, p(x,r)}_{f(x,t)} \right]$$

$$= \int_{-\infty}^{\infty} dr \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \frac{\partial^k}{\partial x^k} \left[\tilde{\Lambda}_t(x,r)\, p(x,t) \right]$$

If $\sum_{k=1}^{\infty} \left| \frac{(-r)^k}{k!} \frac{\partial^k}{\partial x^k} \left[\tilde{\Lambda}_t(x,r)\, p(x,t) \right] \right| < \infty$, then we can exchange sum and integral

$$\Rightarrow \frac{\partial}{\partial t} p(x,t) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial x^k} \left(a_t^{(k)}(x)\, p(x,t) \right)$$

where $a_t^{(k)}(x) \equiv \int_{-\infty}^{\infty} r^k \tilde{\Lambda}_t(x,r)\, dr = \lim_{\Delta t \to 0} \int dx' \, (x-x')^k \, \frac{p(x', t+\Delta t | x,t)}{\Delta t}$

Now it turns out, we can only exchange the sum and the integral if the stochastic paths are continuous functions of time in the sense

$$\forall \epsilon > 0 \quad \lim_{\Delta t \to 0} \int_{|x'-x| > \epsilon} dx \, \frac{p(x', t+\Delta t | x,t)}{\Delta t} = 0 \quad (\text{chance of } x - x' > \epsilon \to 0 \text{ faster than } \Delta t)$$

In that case $a_t^{(k)}(x) = 0 \quad k \geq 3$ (Pawula theorem)

<u>For a Markov process</u>

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} \left(A(x,t) P(x,t) \right) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(B(x,t) P(x,t) \right) \qquad \text{Continuous path}$$

$$= \int dx' \left(\Lambda_t(x|x')\, p(x',t) - \Lambda_t(x'|x)\, p(x,t) \right) \qquad \text{discontinuous paths (jump processes).}$$

Thus the only possible PDE for a Markov process with continuous paths is the general Fokker-Plank equation. Otherwise it is a general master equation (integro differential if $X \in \mathbb{R}$).

<u>Note</u>: this also implies that an equation of motion with third order partial derivatives or higher cannot describe the evolution of a probability distribution with an underlying Markov

process. Equations of motion like this will necessarily lead to unphysical outcomes, such a negative probabilities. This is the case, e.g., for the evolution of the Wigner function with terms beyond the Poisson bracket.

In the multivariate case, with stochastic process $\vec{X}(t)$, $P(\vec{x},t)$ the generally Fokker-Planck equation takes the form

$$\frac{\partial}{\partial t} P(\vec{x},t) = -\sum_i \frac{\partial}{\partial x_i}\left(A_i(\vec{x},t)\, P(\vec{x},t)\right) + \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\left(B_{ij}(\vec{x},t)\, P(\vec{x},t)\right)$$

Where $A_i(\vec{x},t) = \lim_{\Delta t \to 0} \int d\vec{x}'\, (x_i - x_i')\, \frac{P(\vec{x}', t+\Delta t \mid \vec{x}, t)}{\Delta t}$: Drift coefficient

$B_{ij}(\vec{x},t) = \lim_{\Delta t \to 0} \int dx'\, (x_i - x_i')(x_j - x_j')\, \frac{P(\vec{x}', t+\Delta t \mid \vec{x}, t)}{\Delta t}$: Diffusion matrix

Note B_{ij} is positive semidefinite, i.e., has positive eigenvalues at all $\vec{x}$ and t

Examples:

- Hamiltonian evolution: Liouville eqn for $P(\vec{q},\vec{p})$

$$\frac{\partial P}{\partial t} = \{H, P\} = \frac{\partial H}{\partial q_i}\frac{\partial P}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial P}{\partial q_i} = -\sum_i \frac{\partial}{\partial x_i}\left(A_i(\vec{q},\vec{p})\, P(\vec{q},\vec{p})\right)$$

Where $A_i = \begin{bmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q_i} \end{bmatrix}$ is the drift vector.

For Hamiltonian evolution there is no diffusion term $\Rightarrow$ Area in phase space preserved.

- Ornstein-Uhlenbeck Process:

$$\frac{\partial P}{\partial t} = \gamma_i \frac{\partial}{\partial x_i}(x_i P) + \frac{1}{2} D_{ij} \frac{\partial^2}{\partial x_i \partial x_j} P$$

$A_i = -\gamma_i x_i$, $B_{ij} = D_{ij}$ (Diffusion matrix).

Langevin stochastic Equations

We have seen the Orenstein – Uhlenbeck Process yielding the Fokker–Planck equation given on the previous page is described by a Langenvin equation, e.g.,

$$\frac{dp}{dt} = -\frac{\Gamma}{2} p + \sqrt{D}\, \xi(t)$$

where $\xi(t)$ is delta-correlated Gaussian white with

mean $\langle \xi(t) \rangle = 0$, correlation $\langle \xi(t') \xi(t') \rangle = \delta(t-t')$

Integrating

$$p(t) = p(0) - \frac{\Gamma}{2}\int_0^t dt'\, p(t') + \sqrt{D} \underbrace{\int_0^t \xi(t')\,dt'}_{\equiv W(t)}$$

$W(t)$ is in fact a Wiener stochatic variable. To see this note

(1) $W(t)$ is Gaussian since $\xi(t)$ is Gaussian, and thus solely defined by its mean and variance

(2) $\langle W(t) \rangle = \int_0^t \langle \xi(t') \rangle\, dt' = 0$

(3) $\langle W^2(t) \rangle = \langle \Delta W^2(t) \rangle = \int_0^t dt' \int_0^t dt'' \underbrace{\langle \xi(t') \xi(t'') \rangle}_{\delta(t'-t'')} = \int_0^t dt' = t \Rightarrow$ Wiener Process

Note: Since $W(t)$ is nowhere differentiable, $\xi(t) = \frac{dW}{dt}$ doesn't actually exist as a well-defined mathematical quantity. the existence of the delta-function is a dead give away. This means solving this type of differential equation is tricky, involving the tools of "stochastic calculus." A complete discussion of this is beyond the scope of this course. We well touch on the basics. For more details see C. Gardiner "Handbook of Stochastic Methods."

Stochastic differential equations

Let us write the Langenvin equation for the Orenstein – Uhlenbeck equation as

$$dp(t) = -\gamma p(t)\, dt + \sqrt{D}\, dW(t)$$

Where $dW(t) = W(t+dt) - W(t) \sim \xi(t)\, dt$ is a "Wiener increment"

The strange, nondifferentiable nature of the Weiner process implies that the rules of stochastic calculus are different. We will consider the "Itô calculus"

First, consider the finite increment $\Delta W(t) = W(t+\Delta t) - W(t)$

Given the joint probability distribution

$$p(w_n t_n; w_{n-1}, t_{n-1}; \dots w_1 t_1; w_0 t_0) = \prod_{i=0}^{n-1} \underbrace{p(w_{i+1}, t_{i+1} | w_i t_i)}_{\frac{1}{\sqrt{2\pi \Delta t_i}} e^{-\frac{\Delta W(t_i)}{2\Delta t_i^2}}} \; p(w_0, t_0) = \prod_i p(\Delta w_i)\, p(w_0, t_0)$$

$\Rightarrow$ Each increment $\Delta W(t)$ is statistically independent

$\Rightarrow \langle \Delta W(t_1) \Delta W(t_2) \rangle = 0$ if the intervals don't overlap

and $\langle \Delta W(t)^2 \rangle = \Delta t$

We are thus led to the important Îto rule. Let $dw(t) = \lim_{\Delta t \to 0} \Delta W(t)$

$$\boxed{\begin{aligned} \langle dw(t_1)\, dw(t_2) \rangle &= 0 \qquad t_1 \neq t_2 \\ \langle dw(t)^2 \rangle &= dt \end{aligned}}$$

In fact, $dw^2(t) = dt$ for and relatigation of $dw(t)$. Loosely, $dw \sim \sqrt{dt}$, as we saw in the definition of the Wiener process as diffusion under a random walk.

In doing Îto differential stochastic calculus we have thus

$$\boxed{\begin{aligned} dt^2 &= 0 \\ dw\,dt &= 0 \\ dw^2 &= dt \end{aligned}} \quad \text{Itô's rules}$$

The Îto rules mean that we have to be extra careful when compared the normal rules of calculus

Note $\frac{\Delta w}{\Delta t} \sim \frac{1}{\sqrt{\Delta t}}$, which is why this has no finite limit as $\Delta t \to 0$

Consider, then a stochastic variable $X(t)$ which is governed by a generalized Langevin equation

$$\boxed{dx(t) = A(x(t), t)\, dt + B(x(t), t)\, dW(t)} \quad \text{Itô stochastic diff'eq}$$

Numerical integration is tricky, but doable (see

Change of variables: Itô's formula

Consider a function $f(x(t))$ where $X(t)$ satisfies the Itô form of the Langevin equation

$$\Rightarrow df = f(x(t)+dx(t)) - f(x(t))$$
$$= f'(x(t))dx(t) + \frac{1}{2} f''(x(t))\, d^2x(t) \leftarrow \text{need to consider second order}, \quad \text{where } f' = \frac{df}{dx}$$
$$= f'(x(t))(A\,dt + B\,dw) + \frac{1}{2} f''(x(t))\,(A\,dt + B\,dw)^2$$
$$\boxed{df = \left(A(x(t),t)\, f'(x(t)) + \frac{1}{2} B^2(x(t),t)\, f''(x(t))\right)dt + B(x(t),t)\, f'(x(t))\, dW} \quad \text{Itô's rule}$$

In the multivariate case $d\vec{x}(t) = \vec{A}(\vec{x}(t),t)\,dt + \underset{\sim}{B}(\vec{x},t)\,d\vec{w}(t)$, $\langle dw_i(t)\, dw_j(t)\rangle = dt\,\delta_{ij}$

$\uparrow$ Matrix — Independent white noise

$$\Rightarrow \boxed{df = \sum_i \left(A_i(\vec{x}(t),t)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{ij}\left(\underset{\sim}{B}(\vec{x},t)\,\underset{\sim}{B}^T(\vec{x},t)\right)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\right)dt + \sum_{ij} B_{ij}(\vec{x},t)\frac{\partial f}{\partial x_i}\, dW_j(t)}$$

Connection between Stochastic Langevin equation and Fokker-Planck Equation

In the "Hamilton - Langevin" picture

$$\langle f(\vec{x})\rangle(t) = \int p(\vec{x},0)\, f(\vec{x}(t))\, d\vec{x}$$

If Liouville picture

$$\langle f(\vec{x})\rangle(t) = \int p(\vec{x},t)\, f(\vec{x})\, d\vec{x}$$

$$\Rightarrow \langle df\rangle = \int p(x,t)\left[A_i \frac{\partial f}{\partial x_i} + \frac{1}{2}(\underset{\sim}{B}\underset{\sim}{B}^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\right]dt \quad \text{Since } \langle dW_j\rangle = 0$$

Repeated sum convention

Also $\frac{d}{dt}\langle f(\vec{x})\rangle = \frac{\langle df\rangle}{dt} = \int \frac{\partial p}{\partial t} f(\vec{x})\, dt$

Using integration by parts, and since $f(x)$ is arbitrary

$$\Rightarrow \boxed{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x_i}\left(A_i(\vec{x})\, p(x_i,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left(D_{ij}(\vec{x})\, p(\vec{x},t)\right)} \quad \text{Fokker-Planck equation!}$$

where $\vec{A}(\vec{x}) \equiv$ drift vector, $\underset{\sim}{D}(\vec{x}) = \underset{\sim}{B}(\vec{x})\,\underset{\sim}{B}^T(\vec{x}) \geq 0$ Diffusion matrix

In the case of one stochastic variable, given a FP-equation

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(A(x)\, p) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(D(x)\, p)$$

we have a unique associated set of stochastic Langevin equations

$$dx = A(x)\,dt + \sqrt{D(x)}\,dw(t)$$

In the multivariate case, this no unique mapping from FP to Langevin. In the "polar decomposition"

$$\underset{\sim}{D}(\vec{x}) = \underset{\sim}{B}(\vec{x})\,\underset{\sim}{S}(x), \quad \text{where } \underset{\sim}{B}(x) = \sqrt{\underset{\sim}{D}\,\underset{\sim}{D}^T}, \quad \text{and } \underset{\sim}{S}\,\underset{\sim}{S}^T = \mathbb{1} \ (\text{orthogonal})$$

A possible choice is $dx_i = A_i\,dt + B_{ij}\,dw_j$, but any $\underset{\sim}{S}$ will be equivalent.
Typically, one looks to express $\underset{\sim}{D} = \underset{\sim}{B}\,\underset{\sim}{B}^T$, which guarantees it is positive, and then one chooses $\underset{\sim}{B}$ as the weight of the noise term.

Quantum Master equation

Let use return now to the quantum master equation for a single mode boson, with the usual Lindbladian for the damp SHO. In the Wigner-Weyl representation

$$\frac{\partial}{\partial t} W(x,P,t) = \{H, W\}_{MB} + \frac{\Gamma}{2}\left[\frac{\partial}{\partial \alpha}(\alpha W) + \frac{\partial}{\partial \alpha^*}(\alpha^* W)\right] + \Gamma(\bar{n}+\tfrac{1}{2})\frac{\partial^2}{\partial\alpha\,\partial\alpha^*} W \qquad \text{(Complex form)}$$

$$= \{H, W\}_{M.B} + \frac{\Gamma}{2}\left(\frac{\partial}{\partial x}(xW) + \frac{\partial}{\partial P}(PW)\right) + \frac{\Gamma}{2}(\bar{n}+\tfrac{1}{2})\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial P^2}\right) W \qquad \text{(Real variables)}$$

The Lindbladian is the Fokker-Planck equation for an Ornstein-Uhlenbeck process with $\vec{A} = -\frac{\Gamma}{2}\begin{bmatrix} X \\ P \end{bmatrix}$ and diffusion matrix $\underset{\sim}{D} = \frac{\Gamma}{2}(\bar{n}+\tfrac{1}{2})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The Moyal bracket $\{H,W\}_{MB} = \underbrace{-i\left(\frac{\partial H}{\partial\alpha}\frac{\partial W}{\partial\alpha^*} - \frac{\partial H}{\partial\alpha^*}\frac{\partial W}{\partial\alpha}\right)}_{\text{Poisson Bracket}} - \frac{i}{8}\left(\frac{\partial^3 H}{\partial^2\alpha^*\partial\alpha}\frac{\partial^3 W}{\partial^2\alpha\,\partial\alpha^*} - \frac{\partial^3 H}{\partial^2\alpha\partial\alpha^*}\frac{\partial^3 W}{\partial^2\alpha^*\partial\alpha}\right) + \cdots$

The terms corresponding to the Poisson bracket contribute to the drift terms of the FP-equation; they are Hamiltons equations. The higher order terms at minimum are third order partials of W. These are not possible to describe an FP equation for a Markov process, and thus, this is no classical stochastic Langevin equation which describes the underlying trajectories that is why it is a quasiprobability.

In the truncated Wigner approximation (TWA), these higher order terms are neglected. As we saw, this can be a good approximation for the open quantum system, as the diffusion term causes decoherence, which makes the quantum corrections negligible, especially in the macroscopic limit.

Note: The Fokker-Planck description is exact for Gaussian Hamiltonians — Those that map Gaussian states to Gaussian states. The general form is

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\zeta^*\hat{a}^2 + \hbar\zeta\hat{a}^{\dagger 2} \qquad \text{(Quadratic in } \hat{a} \text{ and } \hat{a}^\dagger)$$

Free Rotation: $\hat{H} = \hbar\omega\left(\frac{\hat{x}^2+\hat{p}^2}{2}\right) = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) = \hbar\omega\left(\frac{\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger}{2}\right)$

$H_w(x,P) = \frac{\omega}{2}(x^2+p^2) = \omega\alpha^*\alpha$ (dimensionless units, $\hbar = 1$)

$$\Rightarrow \{H_w, W\} = \omega\left(X\frac{\partial W}{\partial P} - P\frac{\partial W}{\partial X}\right) = \frac{\partial}{\partial x}(-\omega P W) + \frac{\partial}{\partial P}(\omega X W) \Rightarrow \text{Drift } \vec{A} = \begin{pmatrix} \omega P \\ -\omega X \end{pmatrix}$$

$$= -i\omega\left(\alpha^*\frac{\partial W}{\partial\alpha^*} - \alpha\frac{\partial W}{\partial\alpha}\right) = \frac{\partial}{\partial\alpha}(i\omega\alpha W) + \frac{\partial}{\partial\alpha^*}(-i\omega\alpha^* W) \Rightarrow \text{Drift } \vec{A} = \begin{pmatrix} -i\omega\alpha \\ i\omega\alpha^* \end{pmatrix}$$

The diffusion matrix for the single mode

$$D_{XX} = D_{PP} = \Gamma(\bar{n}+\tfrac{1}{2}) \,, \qquad D_{\alpha\alpha^*} = \Gamma(\bar{n}+\tfrac{1}{2})$$

The Hamiltonian evolution is classical flow, defined by the drift terms. Including the damping and diffusion the corresponding Langevin equations are

Damped SHO

$$\boxed{\begin{aligned} dX &= (-\tfrac{\Gamma}{2}X + \omega P)\,dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_X \,, & dP &= (-\tfrac{\Gamma}{2}P - \omega X)\,dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_P \\ d\alpha &= (-i\omega - \tfrac{\Gamma}{2})\alpha\,dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_\alpha \,, & d\alpha^* &= (+i\omega - \tfrac{\Gamma}{2})\alpha^*\,dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_{\alpha^*} \end{aligned}}$$

where $\langle dW_i\, dW_j\rangle = dt\,\delta_{ij}$ for $i,j = X, P$, $dW_\alpha = \frac{dW_X + i\,dW_P}{\sqrt{2}}$, $dW_{\alpha^*} = dW_X - i\,dW_P$

$\langle dW_\alpha^2\rangle = \langle dW_{\alpha^*}^2\rangle = 0 \quad \langle dW_\alpha\, dW_{\alpha^*}\rangle = dt$

The solutions to Langevin equations when averaged over the initial conditions sampled from an initial state yield the solution to the master equation.

The only other exact solution is squeezing. For example

$$H = r\left(\frac{\hat{a}^2 - \hat{a}^{\dagger 2}}{2i}\right) = r\left(\frac{\hat{X}\hat{P} + \hat{P}\hat{X}}{2}\right) \qquad (\hbar = 1)$$

$\Rightarrow H_W(X,P) = \frac{r}{2}(XP + PX) \qquad H_W(\alpha,\alpha^*) = r\left(\frac{\alpha^2 - \alpha^{*2}}{2i}\right)$

$\Rightarrow \{H, W\}_{MB} = \{H, W\}_{PB} = \frac{r}{2}\left(P\frac{\partial H}{\partial P} - X\frac{\partial H}{\partial X}\right)$ (real) $\qquad \{H, W\}_{PB} = -r\left(\alpha\frac{\partial W}{\partial \alpha^*} - \alpha^*\frac{\partial W}{\partial \alpha}\right)$

$\Rightarrow \vec{A} = \frac{r}{2}\begin{bmatrix} X \\ -P \end{bmatrix}$ (real) $\qquad \vec{A} = r\begin{bmatrix} \alpha^* \\ \alpha \end{bmatrix}$ complex

$\Rightarrow$ Langevin equations:
Damped Squeezing

$$\boxed{\begin{aligned} dX &= \left(-\tfrac{\Gamma}{2}X - \tfrac{r}{2}X\right)dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_X & dP &= \left(-\tfrac{\Gamma}{2}P + \tfrac{r}{2}P\right)dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_P \\ d\alpha &= -r\alpha^*\,dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_\alpha \,, & d\alpha^* &= -r\alpha\,dt + \sqrt{\Gamma(\bar{n}+\tfrac{1}{2})}\,dW_{\alpha^*} \end{aligned}}$$

The Bogliubov interaction "squeezes" X and "antisqueezes" P (with the particular choice of phase and $r>0$). The damping, shrinks both X & P ; the diffusion adds fluctuations to X & P.

Other Phase-Space Representations

While we have focused on the Wigner representation, we have seen that there are other representations, e.g., associated with other operator orderings. For example, for the anharmonic oscillator

$$\hat{H} = \frac{\hbar\kappa}{2}\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \quad \text{(Kerr effect)}, \quad \text{including the usual damping,}$$

as we show in homework, the time evolution of the Husimi Q-function is

$$\frac{\partial}{\partial t}Q = +i\kappa\left(\frac{\partial}{\partial\alpha}(|\alpha|^2-1)\alpha Q - \frac{\partial}{\partial\alpha^*}(|\alpha|^2-1)\alpha^* Q\right) + i\kappa\left(\frac{\partial^2}{\partial\alpha^2}\alpha^2 Q - \frac{\partial^2}{\partial\alpha^*}\alpha^{*2}Q\right)$$
$$+\Gamma\left(\frac{\partial}{\partial\alpha}(\alpha Q) + \frac{\partial^2}{\partial\alpha^*}(\alpha^* Q)\right) + \Gamma(\bar{n}+1)\frac{\partial^2 Q}{\partial\alpha\,\partial\alpha^*}$$

Unlike for the Wigner function, the equation of motion for the Husimi distribution has no third-order partials. With no-more than two-body interactions, the PDE is second order. So it has the form of a Fokker-Planck equation! Here the diffusion matrix (complex) $D_{\alpha\alpha} = 2i\kappa\alpha^2 = (D_{\alpha^*\alpha^*})^*$, $D_{\alpha\alpha^*} = \Gamma(\bar{n}+1)$

But wait a minute. We know that the Kerr system has a negative Wigner function, and thus cannot be described by classical statistical theory. A change of representation cannot change this fact. So where is the quantumness hiding?

The tricky answer is that generally, for the anharmonic oscillator the diffusion matrix has negative eigenvalues. Thus it cannot be describable as a classical ensemble of realistic trajectories evolving according to Stochastic Langevin equations. Note, however, in the presence of decoherence the positivity of the diffusion matrix can be restored! This is another example of how decoherence restores classicality.