

Phys 581 - Open Quantum Systems

Lecture 17: The Stochastic Schrödinger Equation: Quantum State Diffusions

Formal description of Jump (Poisson) Process

We have seen that the Lindblad master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \sum_{\mu} \hat{L}_{\mu} \hat{\rho} \hat{L}_{\mu}^{\dagger}, \quad \text{where} \quad \hat{H}_{\text{eff}} = \hat{H} - \frac{i\hbar}{2} \sum_{\mu} \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu}$$

is formal equivalent to the ensemble average over all quantum trajectories

$$\hat{\rho}(t) = \langle\langle |\psi(t)\rangle\langle\psi(t)| \rangle\rangle$$

Where $|\psi(t)\rangle$ evolves stochastically according to

$$|\psi(t+dt)\rangle \Rightarrow \begin{cases} \frac{\hat{L}_{\mu} |\psi\rangle}{\|\hat{L}_{\mu} |\psi\rangle\|} & \text{with probability } dp_{\mu}(t) = \langle\psi(t)| \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} |\psi(t)\rangle dt \\ \frac{(1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt) |\psi(t)\rangle}{\|(1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt) |\psi(t)\rangle\|} & \text{with probability } p_0(t) = 1 - \sum_{\mu} dp_{\mu}(t) \end{cases}$$

We can formally express the evolution of $|\psi(t)\rangle$ as a "stochastic differential equation"

$$|\psi(t+dt)\rangle = \left(1 - \sum_{\mu} dN_{\mu}\right) \frac{(1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt) |\psi(t)\rangle}{\|(1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt) |\psi(t)\rangle\|} + \sum_{\mu} dN_{\mu}(t) \frac{\hat{L}_{\mu} |\psi\rangle}{\|\hat{L}_{\mu} |\psi\rangle\|}$$

where $dN_{\mu}(t)$ is a "stochastic interval", taking on random values

$$dN_{\mu}(t) = \begin{cases} 1 & \text{with probability } dp_{\mu}(t) = \langle\psi(t)| \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} |\psi(t)\rangle dt \\ 0 & \text{with probability } 1 - dp_{\mu}(t) \end{cases}$$

the intervals at different times are uncorrelated, making this a "Poisson process" associated with counting statistics. From these it follows

$$\langle\langle dN_{\mu} \rangle\rangle = dp_{\mu} = \langle\psi(t)| \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} |\psi(t)\rangle dt$$

$$dN_{\mu} dN_{\nu} = dN_{\mu} \delta_{\nu\mu}$$

To rule of stochastic calculus is then to keep term to $\mathcal{O}(dt)$. Since $dN = \mathcal{O}(dt)$
 $dN dt = 0$

The stochastic equation is then

$$|\psi(t+dt)\rangle = \frac{(1 - \frac{i}{\hbar} dt \hat{H}_{\text{eff}}) |\psi(t)\rangle}{\sqrt{\langle \psi(t) | 1 - \frac{i}{\hbar} (\hat{H}_{\text{eff}} - \hat{H}_{\text{eff}}) dt | \psi(t) \rangle}} + \sum_{\mu} dN_{\mu} \left[\frac{\hat{L}_{\mu}}{\sqrt{\langle \psi(t) | \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} | \psi(t) \rangle}} - 1 \right] |\psi(t)\rangle$$

$$= (1 - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt + \frac{1}{2} \sum_{\mu} \langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle dt) |\psi(t)\rangle + \sum_{\mu} dN_{\mu} \left[\frac{\hat{L}_{\mu}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - 1 \right] |\psi(t)\rangle$$

$$\Rightarrow |d\psi(t)\rangle = |\psi(t+dt)\rangle - |\psi(t)\rangle$$

$$= \left(-\frac{i}{\hbar} \hat{H}_{\text{eff}} + \frac{1}{2} \sum_{\mu} \langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle \right) dt |\psi\rangle + \sum_{\mu} \left[\frac{\hat{L}_{\mu}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - 1 \right] dN_{\mu} |\psi\rangle$$

this is a form of the **Stochastic Schrödinger Equation (SSE)**. It is stochastic because of the stochastic interval dN_{μ} . It is also nonlinear, as expected, through the renormalization. Note we can write this formally as

$$\frac{d}{dt} |\psi(t)\rangle = \left(-\frac{i}{\hbar} \hat{H}_{\text{eff}} + \frac{1}{2} \sum_{\mu} \langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle \right) |\psi(t)\rangle + \sum_{\mu} \left[\frac{\hat{L}_{\mu}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - 1 \right] \frac{dN_{\mu}}{dt} |\psi(t)\rangle$$

where $\frac{dN_{\mu}}{dt}$ = Current of random counts.

We show that ^{the} SSE, when ensemble averaged, yields the Lindblad Master Equation:

$$d\rho = \langle\langle d|\psi\rangle\langle\psi| \rangle\rangle = \langle\langle |\psi\rangle\langle d\psi| + |d\psi\rangle\langle\psi| + \underbrace{|d\psi\rangle\langle d\psi|}_{\neq} \rangle\rangle$$

In the stochastic calculus, we must be careful to keep terms of order dt

$$\begin{aligned}
\Rightarrow d\hat{\rho} &= \left\langle \left\langle \frac{-i\hat{H}_{\text{eff}} dt}{\hbar} |\psi\rangle\langle\psi| + \frac{i}{\hbar} |\psi\rangle\langle\psi| \hat{H}_{\text{eff}} dt + \sum_{\mu} \langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle dt |\psi\rangle\langle\psi| \right\rangle \right. \\
&\quad \left. + \sum_{\mu} \left\langle \left\langle dN_{\mu} \left(\frac{\hat{L}_{\mu}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - 1 \right) |\psi\rangle\langle\psi| + |\psi\rangle\langle\psi| \left(\frac{\hat{L}_{\mu}^{\dagger}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - 1 \right) \right\rangle \right\rangle \right. \\
&\quad \left. + \sum_{\mu} \left\langle \left\langle dN_{\mu} \left(\frac{\hat{L}_{\mu}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - 1 \right) |\psi\rangle\langle\psi| \left(\frac{\hat{L}_{\mu}^{\dagger}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - 1 \right) \right\rangle \right\rangle \right. \\
&= \frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] dt + \sum_{\mu} \langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle dt \hat{\rho} + \sum_{\mu} \langle \langle dN_{\mu} \rangle \rangle \left(\frac{\hat{L}_{\mu} \hat{\rho} \hat{L}_{\mu}^{\dagger}}{\sqrt{\langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle}} - \hat{\rho} \right)
\end{aligned}$$

Treated dN_{μ} and $|\psi\rangle\langle\psi|$ as statistically independent $\langle \langle dN_{\mu} |\psi\rangle\langle\psi| \rangle \rangle = \langle \langle dN_{\mu} \rangle \rangle \hat{\rho}$ (gives the right answer for the wrong reason - see D. Stuck "Quantum Optics")

$$\text{Now } \langle \langle dN_{\mu} \rangle \rangle = \langle \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rangle dt$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = \frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \sum_{\mu} \hat{L}_{\mu} \hat{\rho} \hat{L}_{\mu}^{\dagger} \quad \checkmark \quad \text{q.e.d.}$$

Weiner Process and Quantum State Diffusion

We have seen that we obtain an equivalence class of "unravellings" of the master equation with a unitary remixing of the Krause operators $\hat{M}_{\mu} = \hat{L}_{\mu} \sqrt{dt}$, $\mu=1, \dots, m$. This is a limited class of remixings. A more general equivalence is found including $\hat{M}_0 = \mathbb{1} - \frac{i}{\hbar} \hat{H}_{\text{eff}} dt$.

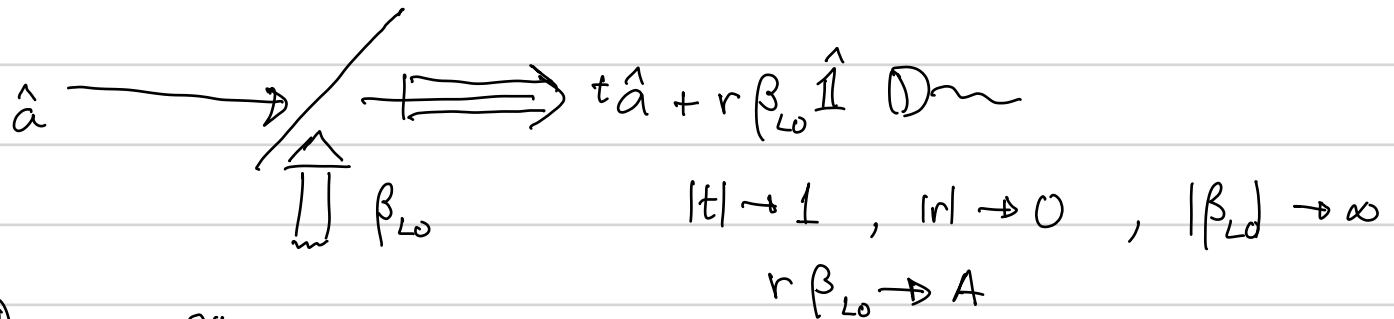
Consider a master equation with one Lindblad operator \hat{L} , and thus two Kraus operators $\hat{M}_1 = \hat{L} \sqrt{dt}$, $\hat{M}_0 = \mathbb{1} - \frac{i}{\hbar} \hat{H} dt - \frac{1}{2} \hat{L}^{\dagger} \hat{L} dt$. We can define a unitary remixing

$$\begin{bmatrix} \hat{N}_0 \\ \hat{N}_1 \end{bmatrix} = \begin{bmatrix} \mathbb{1} - \frac{1}{2} A^{\dagger} A dt & -A^* \sqrt{dt} \\ A \sqrt{dt} & \mathbb{1} - \frac{1}{2} A A^{\dagger} dt \end{bmatrix} \begin{bmatrix} \hat{M}_0 \\ \hat{M}_1 \end{bmatrix}$$

$$\begin{aligned}
\Rightarrow \hat{N}_1 &= \underbrace{(\hat{L} + A)}_{\hat{J}} \sqrt{dt}, & \hat{N}_0 &= \mathbb{1} - \frac{1}{2} A^{\dagger} A dt - \frac{i}{\hbar} \hat{H} dt - \frac{1}{2} \hat{L}^{\dagger} \hat{L} dt - A^* \hat{L} dt \\
& & &= \mathbb{1} - \frac{i}{\hbar} \tilde{H} dt - \frac{1}{2} (\hat{L} + A)^{\dagger} (\hat{L} + A) dt + \frac{1}{2} (A \hat{L}^{\dagger} - A^* \hat{L}) dt \\
& & &= \mathbb{1} - \frac{i}{\hbar} \tilde{H} dt - \frac{1}{2} \hat{J}^{\dagger} \hat{J} dt
\end{aligned}$$

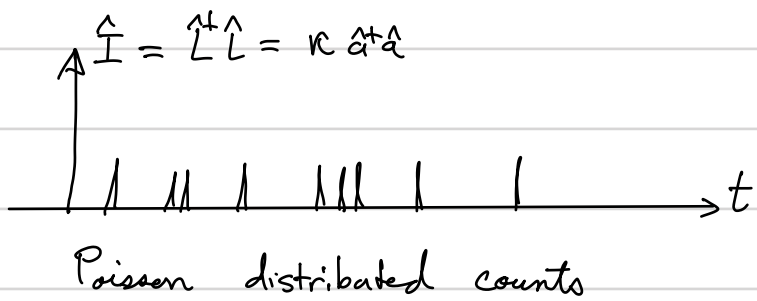
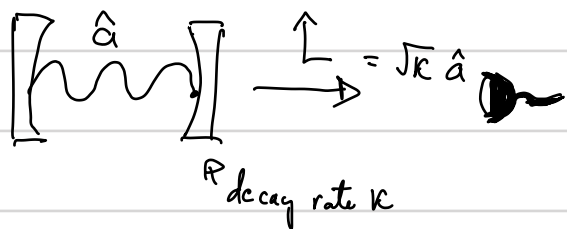
New jump operator $\hat{J} = \hat{L} + A$, New Hamiltonian $\tilde{H} = \hat{H} - \frac{i\hbar}{2} (A^* \hat{L} - A \hat{L}^{\dagger})$

We can ^{always} interpret a particular set of jump operators (and thus unravelling of the master equation) as associated with a particular measurement done on the environment. We recognize this jump operator as corresponding to unbalanced homodyne detection.



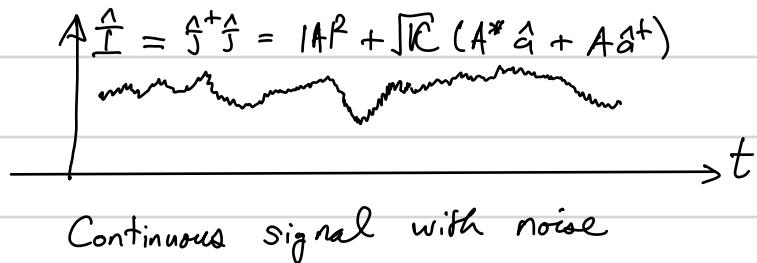
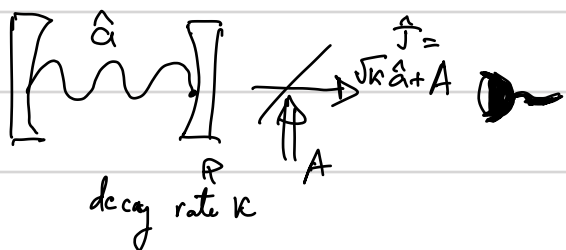
For example: Decaying SHO

Direct Detection



$\hat{L}^\dagger \hat{L} = \kappa \hat{a}^\dagger \hat{a} = \text{Flux of output photons (rate)}$

Homodyne Detection



$|A|^2 = \text{Flux of local oscillator}, |A| \rightarrow \infty$: Macroscopic # of photons in time $\frac{1}{\kappa}$
 Define $A = \sqrt{\kappa} \alpha, |\alpha|^2 \gg 1$

In order to make the transition from the discrete Poisson process to continuous noise, we "coarse grain" the quantum trajectory. Consider a coarse-grained time interval δt

$$\frac{1}{|A|^2} = \frac{1}{\kappa |\alpha|^2} \ll \delta t \ll \frac{1}{\kappa}$$

In this time interval the $\langle\langle \delta N \rangle\rangle = \langle \hat{J}^\dagger \hat{J} \rangle \delta t = |A|^2 \delta t + \langle A \hat{L}^\dagger + A^* \hat{L} \rangle \delta t + \langle \hat{L}^\dagger \hat{L} \rangle \delta t \gg 1$

In the limit of large mean, the Poisson distribution is well approximated by a Gaussian with fluctuations about mean $\langle\langle (\delta N - \langle\langle \delta N \rangle\rangle)^2 \rangle\rangle = \langle\langle \delta N \rangle\rangle$. This can be expressed in the stochastic calculus in terms of a "Wiener process" corresponding to a Gaussian random variable whose variance grows like t , as in a random walk.

Keeping terms of to order $|A|$

$$\delta N \rightarrow [|A|^2 + |A| \langle \hat{X}_\phi \rangle] \delta t + |A| \delta W(t), \quad \hat{X}_\phi = e^{-i\phi} \hat{L} + e^{i\phi} \hat{L}^\dagger$$

Where $\delta W(t)$ is a Wiener interval, Gaussian random variable with mean 0 and variance δt : $\langle \delta W(t) \rangle = 0$, $\langle \delta W(t)^2 \rangle = \delta t$. Thus $\delta W(t) \sim \sqrt{\delta t}$.

In the stochastic calculus, $\delta W \rightarrow dW$. One can show

$$\langle dW(t) \rangle = 0 \quad dW^2 = dt \quad \forall \text{ cases; no need to ensemble average.}$$

The current seen by the photo-detector $I(t) = \langle I \rangle + \xi(t)$, where $\xi(t) = |A| \frac{dW}{dt}$

One can show $\langle \xi(t) \xi(t') \rangle = |A|^2 \delta(t-t')$: Delta-correlated, Langevin

Our stochastic differential equation for this continuous current is.

$$d|\psi\rangle = \left[\frac{i}{\hbar} \tilde{H} dt - \frac{1}{2} (\hat{J}^\dagger \hat{J} - \langle \hat{J}^\dagger \hat{J} \rangle) dt \right] |\psi(t)\rangle + \left[\frac{\hat{J}}{\|\hat{J}|\psi\rangle\|} - 1 \right] dN |\psi(t)\rangle$$

Aside: $\frac{\hat{J}|\psi\rangle}{\|\hat{J}|\psi\rangle\|} = \frac{(\hat{L}+A)|\psi\rangle}{\|\hat{L}+A\|} = \frac{e^{-i\phi} \hat{L} + |A|}{\|(e^{-i\phi} \hat{L} + |A|)|\psi\rangle\|}$ where $A = |A|e^{i\phi}$
 (more convenient phase choice)

$$= \frac{e^{-i\phi} \hat{L} + A}{\sqrt{|A|^2 + |A| \langle \hat{X}_\phi \rangle + \langle \hat{L}^\dagger \hat{L} \rangle}} = (1 + e^{-i\phi} \frac{\hat{L}}{|A|}) (1 + \frac{\langle \hat{X}_\phi \rangle}{|A|} + \langle \hat{L}^\dagger \hat{L} \rangle)^{-1/2} \approx (1 + e^{-i\phi} \frac{\hat{L}}{|A|}) (1 - \frac{1}{2} \frac{\langle \hat{X}_\phi \rangle}{|A|} + \frac{3}{8} \frac{\langle \hat{X}_\phi \rangle^2}{|A|^2} - \frac{1}{2} \langle \hat{L}^\dagger \hat{L} \rangle)$$

$$\approx 1 + e^{-i\phi} \frac{\hat{L}}{|A|} - \frac{1}{2} \frac{\langle \hat{X}_\phi \rangle}{|A|} + \frac{3}{8} \frac{\langle \hat{X}_\phi \rangle^2}{|A|^2} - \frac{1}{2} e^{-i\phi} \frac{\hat{L} \langle \hat{X}_\phi \rangle}{|A|^2} - \frac{1}{2} \frac{\langle \hat{L}^\dagger \hat{L} \rangle}{|A|^2}, \text{ where } \hat{X}_\phi = e^{i\phi} \hat{L} + e^{-i\phi} \hat{L}^\dagger$$

$$\Rightarrow \left[\frac{\hat{J}}{\|\hat{J}|\psi\rangle\|} - 1 \right] dN \approx \left[e^{-i\phi} \frac{\hat{L}}{|A|} - \frac{1}{2} \frac{\langle \hat{X}_\phi \rangle}{|A|} + \frac{3}{8} \frac{\langle \hat{X}_\phi \rangle^2}{|A|^2} - \frac{1}{2} e^{-i\phi} \frac{\hat{L} \langle \hat{X}_\phi \rangle}{|A|^2} - \frac{1}{2} \frac{\langle \hat{L}^\dagger \hat{L} \rangle}{|A|^2} \right] * [|A|^2 dt + |A| \langle \hat{X}_\phi \rangle dt + |A| dW]$$

Dropping terms $\mathcal{O}(\frac{1}{|A|})$ and smaller

$$\left[\frac{\hat{J}}{\|\hat{J}|\psi\rangle\|} - 1 \right] dN \approx A^* \hat{L} dt - \frac{1}{2} |A| \langle \hat{X}_\phi \rangle dt + \frac{1}{2} e^{-i\phi} \hat{L} \langle \hat{X}_\phi \rangle dt - \frac{1}{8} \langle \hat{X}_\phi \rangle^2 dt - \frac{1}{2} \langle \hat{L}^\dagger \hat{L} \rangle dt + (e^{-i\phi} \hat{L} - \frac{1}{2} \langle \hat{X}_\phi \rangle) dW$$

Aside: $-\frac{i}{\hbar} \hat{H} dt - \frac{1}{2} (\hat{J}^\dagger \hat{J} - \langle \hat{J}^\dagger \hat{J} \rangle) dt =$

$$= -\frac{i}{\hbar} \hat{H} dt - \frac{1}{2} (A^* \hat{L} - A \hat{L}^\dagger) dt - \frac{1}{2} (\hat{L}^\dagger \hat{L} + A^* \hat{L} + A \hat{L}^\dagger - \langle \hat{L}^\dagger \hat{L} \rangle - |A| \langle \hat{X}_\phi \rangle) dt$$

$$= -\frac{i}{\hbar} \hat{H}_{\text{eff}} dt - A^* \hat{L} dt - \frac{1}{2} \hat{L}^\dagger \hat{L} dt + \frac{1}{2} \langle \hat{L}^\dagger \hat{L} \rangle dt + \frac{1}{2} |A| \langle \hat{X}_\phi \rangle dt$$

Putting this all together:

$$d|\psi\rangle = \left[-\frac{i}{\hbar} \hat{H} - \frac{1}{2} (\hat{L}^\dagger \hat{L} - \langle \hat{X}_\phi \rangle \hat{L} e^{-i\phi} + \frac{1}{4} \langle \hat{X}_\phi \rangle^2) \right] dt |\psi\rangle + \left[e^{-i\phi} \hat{L} - \frac{1}{2} \langle \hat{X}_\phi \rangle \right] dW |\psi(t)\rangle$$

This is the Stochastic Schrödinger equation for quantum trajectories associated with homodyne detection.

Check: When ensemble-averaged, we recover the Lindblad master equation

$$d\hat{\rho} = \langle\langle d|\psi\rangle\langle\psi| \rangle\rangle = \langle\langle |\psi\rangle\langle\psi| + |\psi\rangle\langle d\psi| + |d\psi\rangle\langle\psi| \rangle\rangle$$

(Note: For a Wiener process $\langle\langle dW(t) |\psi(t)\rangle\langle\psi(t)| \rangle\rangle = \langle\langle dW(t) \rangle\rangle \langle\langle |\psi(t)\rangle\langle\psi(t)| \rangle\rangle = 0$)

$$\Rightarrow d\hat{\rho} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] dt + \frac{1}{2} \langle \hat{X}_\phi \rangle (e^{i\phi} \hat{L} \hat{\rho} + \hat{\rho} \hat{L}^\dagger e^{-i\phi}) dt - \frac{1}{4} \langle \hat{X}_\phi \rangle^2 \hat{\rho} dt + \underbrace{\left(e^{-i\phi} \hat{L} - \frac{1}{2} \langle \hat{X}_\phi \rangle \right) \hat{\rho} \left(e^{i\phi} \hat{L}^\dagger - \frac{1}{2} \langle \hat{X}_\phi \rangle \right)}_{\frac{d\hat{\rho}}{dt}} dW^2$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \hat{L} \hat{\rho} \hat{L}^\dagger, \quad \text{where } \hat{H}_{\text{eff}} = \hat{H} - \frac{i\hbar}{2} \hat{L}^\dagger \hat{L}$$

This is a form known as Quantum State Diffusion.

To see why this is so, consider the case where $\phi=0$, $\hat{L} = \hat{L}^\dagger \Rightarrow \hat{X} = 2\hat{L}$

$$\Rightarrow d|\psi\rangle = \underbrace{\left[-\frac{i}{\hbar} \hat{H} - \frac{1}{2} (\hat{L} - \langle \hat{L} \rangle)^2 \right]}_{\text{drift}} dt |\psi\rangle + \underbrace{(\hat{L} - \langle \hat{L} \rangle)}_{\text{diffusion}} dW |\psi\rangle$$

$$\frac{d|\psi\rangle}{dt} = \left(-\frac{i}{\hbar} \hat{H} - \frac{1}{2} (\hat{L} - \langle \hat{L} \rangle)^2 \right) |\psi\rangle + (\hat{L} - \langle \hat{L} \rangle) |\psi\rangle \xi(t)$$

The SSE looks like a Langevin equation with a term that tries to damp towards an eigenstate of \hat{L} , in which case $(\hat{L} - \langle \hat{L} \rangle) |\psi\rangle = 0$, with Langevin noise to satisfy the "fluctuation-dissipation theorem. Note if $[\hat{H}, \hat{L}]$, the

\hat{H} and \hat{L} share common eigenstates. The system will dynamically evolve into an eigenstate of \hat{L} . This is a kind of "dynamical collapse of the wavefunction" under continuous measurement of the observable \hat{L} . This kind of measurement is known as a "quantum non-demolition measurement" (QND) for historical reasons.

The master equation is recovered from the ensemble average of quantum trajectories. Let $\Delta\hat{L} \equiv \hat{L} - \langle\hat{L}\rangle$: $d|\psi\rangle = \left(\frac{-i\hat{H}}{\hbar} - \frac{1}{2}\Delta\hat{L}^2\right)|\psi\rangle dt + \Delta\hat{L}|\psi\rangle dW$

$$d\rho = \langle\langle d(|\psi\rangle\langle\psi|)\rangle\rangle = \frac{-i}{\hbar}[\hat{H}, \rho]dt - \frac{1}{2}(\Delta\hat{L}^2\rho + \rho\Delta\hat{L}^2)dt + \Delta\hat{L}\rho\Delta\hat{L}\langle\langle dW^2\rangle\rangle + (\Delta\hat{L}\rho + \rho\Delta\hat{L})\langle\langle dW\rangle\rangle$$

$\langle\langle dW^2\rangle\rangle = dt$ $\langle\langle dW\rangle\rangle = 0$

$$\Rightarrow \frac{d\rho}{dt} = \frac{-i}{\hbar}[\hat{H}, \rho] - \frac{1}{2}[\Delta\hat{L}, [\Delta\hat{L}, \rho]] = \frac{-i}{\hbar}[\hat{H}, \rho] - \frac{1}{2}[\hat{L}, [\hat{L}, \rho]]$$

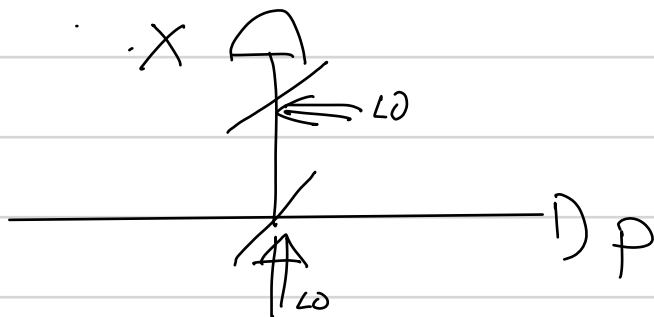
This is the Lindblad form of the master equation when $\hat{L} = \hat{L}^\dagger$, noting that $[\hat{L}, [\hat{L}, \rho]] = \hat{L}(\hat{L}\rho - \rho\hat{L}) - (\hat{L}\rho - \rho\hat{L})\hat{L} = \hat{L}^2\rho + \rho\hat{L}^2 - 2\hat{L}\rho\hat{L}$

The idea of Quantum State Diffusion and dynamical collapse was first studied by Aisin and Percival (Phys. Rev. Lett. 52 1657 (1984), Helvetica Phys. Acta 62, 363 (1989); Phys. Lett. A 143, 1 (1990); Phys. Lett. A 175 144 (1993)). The S.S.E. they wrote is

$$d|\psi\rangle = \frac{-i}{\hbar}\hat{H}dt|\psi\rangle - \frac{1}{2}(\hat{L}^\dagger\hat{L} - 2\langle\hat{L}^\dagger\rangle\hat{L} + |\langle\hat{L}\rangle|^2)dt|\psi\rangle + (\hat{L} - \langle\hat{L}\rangle)dW_\alpha|\psi\rangle$$

α Complex Wiener

This is another form of quantum-state diffusion. This unravelling of the master equation corresponds to heterodyne detection, as opposed to homodyne detection. In heterodyne, instead of measure a quadrature \hat{X}_ϕ , one measures both \hat{X} and \hat{P}



This corresponds to a "complex" photon current $\langle\hat{X}\rangle + i\langle\hat{P}\rangle = \langle\hat{a}\rangle$

Note when $\hat{L} = \hat{L}^\dagger$ $d|\psi\rangle = \frac{-i}{\hbar}\hat{H}dt|\psi\rangle - \frac{1}{2}(\hat{L} - \langle\hat{L}\rangle)^2dt|\psi\rangle + (\hat{L} - \langle\hat{L}\rangle)\frac{(dW_x + i dW_p)}{\sqrt{2}}dt|\psi\rangle$ as in the homodyne case.