Physics 581: Open Quantum Systems
Problem Set #1
Due Tuesday Feb. 9, 2023

Problem 1: Different ensemble decompositions - example, spin ½ (15 points)
(a) Suppose we have a statistical mixture of spin 1/2 particles that consists of the state $|\uparrow_z\rangle$ mixture with probability $\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$ and the state $|\downarrow_z\rangle$ with probability $\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$.

Find the matrix of the density operator in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$, and in the basis of eigenstates of $\hat{\sigma}_x$, $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$. What is the Bloch vector that describes this state?

(b) Now suppose we have a mixed state with 1/2 probability to have spin along $\mathbf{e}_{n_1} = \frac{1}{\sqrt{2}}(\mathbf{e}_z + \mathbf{e}_x)$ and 1/2 probability to have spin along $\mathbf{e}_{n_2} = \frac{1}{\sqrt{2}}(\mathbf{e}_z - \mathbf{e}_x)$. Is this a completely mixed state? Write the density operator in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$. Compare to part (b). Please comment on your result.

(c) Show that two statistical mixtures of pure states, $\{|\uparrow_n\rangle\}$ with probabilities p_n, and $\{|\uparrow_m\rangle\}$ with probabilities q_m, describe the *same* density operator $\hat{\rho}$ if

$$\mathbf{Q} = \sum_n p_n \mathbf{e}_n = \sum_m q_m \mathbf{e}_m \, ,$$

where **Q** is the Bloch vector of $\hat{\rho}$. Check this with your results of parts (b) and (c).

Problem 2: Ambiguity of ensemble decompositions of density operators (15 points)
We saw in Problem 1 that a density operator does not decompose uniquely into a statistical mixture of pure states. This has profound implications for both practical calculations of the density matrix (as we will see later in the semester) as well as for foundational descriptions of states in quantum mechanics. What different ensemble are possible to yield a given density operator? In this problem we prove the following.

<u>Schrödinger- Hughston-Jozsa-Wootters (HJW) theorem:</u>

The two density operators

$$\hat{\rho}_1 = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i| \qquad \hat{\rho}_2 = \sum_{j=1}^{l} q_j |\phi_j\rangle\langle\phi_j|$$

are equal if and only if the two ensembles are related by,

$$\sqrt{q_j}|\phi_j\rangle = \sum_{i=1}^{k} U_{ji}\sqrt{p_i}|\psi_i\rangle$$

where U_{ji} are elements of an isometry matrix (their rows and columns are orthogonal).

(a) Assume the relation between the ensembles is true. Prove that $\hat{\rho}_1 = \hat{\rho}_2$.

(b) Assume $\hat{\rho}_1 = \hat{\rho}_2 \equiv \hat{\rho}$. Show $\sqrt{q_j}\left|\phi_j\right\rangle = \sum_i U_{ji}\sqrt{p_i}\left|\psi_i\right\rangle$.

(Hint: Show first that $\sqrt{p_i}\left|\psi_i\right\rangle = \sum_\alpha M_{j\alpha}\sqrt{\lambda_\alpha}\left|e_\alpha\right\rangle$, where λ_α are the eigenvalues of $\hat{\rho}$ and $\left|e_\alpha\right\rangle$ its orthonormal eigenvectors and $M_{j\alpha}$ are elements of a unitary matrix. The same thus holds for $\sqrt{q_j}\left|\phi_j\right\rangle$. The proof will follow).

Historical note: "The theorem was originally proven by Schrödinger in 1936. He commented that this theorem was one 'for which I claim no priority but the permission of deducing it in the following section, for it is certainly not well known.' His comment was amusingly prescient: The theorem was rediscovered by Jaynes in 1957 (whose work was extended by Hadjisavvas (1981)), rediscovered by Hughston, Jozsa, and Wootters (HJW) in 1993 (this last an expansion of a 1989 partial rediscovery by Gisin); in 1999, Mermin simplified a portion of HJW's proof - and it would appear none of these were aware of Schrödinger's work. Furthering the irony, Mermin commented that this is 'a pertinent theorem which deserves to be more widely known.' "

Problem 3: Inhomogeneous broadening (25 Points)

Rabi oscillations will decay because the two-level system is an open quantum system; coherence decays on a time scale we denote T_2. The oscillations can also decay because of uncertainty in the parameters of Hamiltonian. In modern parlance, we might call these "coherent errors." In particular, when we are measuring an expectation value, we must average over a large ensemble of measurement outcomes. If the value of the Rabi frequency and/or detuning are different for different members of ensemble, this is known "inhomogeneous broadening." This can be true because we have an extended ensemble in space and the fields that define the Hamiltonian are spatially inhomogeneous across the ensemble, or for a single system reprepared and measured, the field varies from shot-to-shot. We study in this problem the dephasing of oscillations on a time scale known as T_2^*, and explore the difference from true decay due to decoherence.

(a) Consider an ensemble of spins undergoing Rabi oscillations (spin magnetic resonance) in a bias static magnetic field *B*, giving a resonance frequency $\omega_0 = \gamma B$ and driven by an rf-magnetic field transversely oscillating at frequency ω. The cloud is extended so that spins see an inhomogeneous distribution of bias B-fields, which we will take to be Gaussian distributed. Because of this, the detuning from resonance, $\Delta = \omega - \omega_0$, will be inhomogeneous according to a probability distribution

$$P(\Delta) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}}$$

where Δ_0 is the average detuning and δ^2 is the variance. The measured Rabi oscillation is the average across the ensemble is

$$P(\uparrow_z, t) = \int_{-\infty}^{\infty} d\Delta \; P(\Delta) P(\uparrow_z, t|\Delta, \Omega)$$

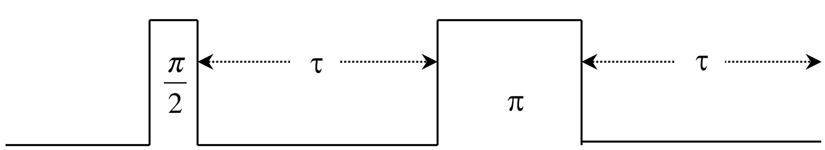
where $P(\uparrow_z, t|\Delta, \Omega)$ is the probability of finding spin-up for a fixed Rabi frequency and detuning (assume initially spin down). Show that for very small inhomogeneity, $\delta \ll \Omega, |\Delta_0|$

$$P(\uparrow_z, t) \approx \frac{\Omega^2}{2\Omega_{tot}^2} \left[1 - \cos(\Omega_{tot} t) \exp\left(-\frac{t^2}{2T_2^{*2}}\right)\right]$$, where $\Omega_{tot} = \sqrt{\Omega^2 + \Delta_0^2}$.

$1/T_2^* = \delta|\Delta_0|/\Omega_{tot}$ is the inhomogeneous linewidth, resulting from dephasing on the timescale of T_2^*. What is this linewidth in the limiting cases, $|\Delta_0|/\Omega \ll 1$ and $|\Delta_0|/\Omega \gg 1$?

(c) Plot this solution for different ratios of δ/Ω, taking $|\Delta_0|/\Omega = 1$.

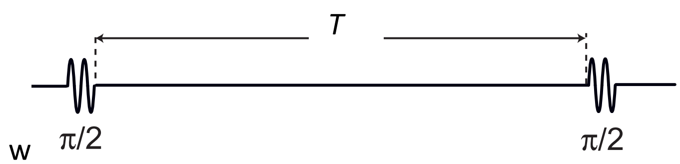
(d) Spin echo: Though inhomogeneous broadening will cause a decay of the ensemble averaged coherence; it is not a truly irreversible process. A way to see this is through the phenomenon of "spin echo". Consider an ensemble of spins sitting in the inhomogeneous bias B-field, as in part (a). Now consider the following pulse sequence.



The $\pi/2$-pulse about the *x*-axis to bring all spins onto the *y*-axis of the Bloch sphere. For a time τ, the spins randomly precess about the z-axis and the ensemble dephases. The π-pulse about the *x*-axis acts to time reverse the process. An "echo" signal will be seen at a time τ later when the spins "refocus" and returns to its initial. **Explain this process using this Bloch sphere.**

Problem 4: Ramsey fringes and the measurement of T2 times (25 Points)

(a) We seek to measure the coherence of between the computational basis states of a qubit $\{|0\rangle,|1\rangle\}$. Consider a two-pulse Ramsey sequence: A fast $\pi/2$ pulse around *x*-axis with detuning Δ, free evolution for a time T, a second fast $\pi/2$ pulse around *x*-axis at the same detuning Δ.



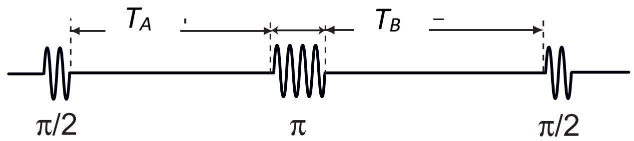
During the free evolution, in the absence of decay, the qubit will precess around the z-axis of the Bloch sphere. In the rotating frame, it precess at the frequency $\omega_0-\omega=-\Delta$, where w is the frequency of the driving pulse, and ω_0 is the qubit resonance frequency. In reality, the coherence ρ_{01} decays exponentially with rate $1/T_2$. Show that, given the qubit initially in $|1\rangle$, the probability to find $|0\rangle$ after the sequence is

$$P_0 = \frac{1}{2}\left[1+\cos(\Delta T)e^{-T/T_2}\right]$$

Explain this using the evolution on the Bloch sphere. Plot this for $\Delta/2\pi = 1$ MHz and T_2=25 μs, for T=0 to 25 μs.

(b) Suppose now that in addition to homogeneous decay, there is inhomogeneous decay T_2^*. Suppose that if the pulses are tuned to frequency ω, the probability the detuning seen by the qubit is Gaussian distributed, $p(\Delta) = e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}} / \sqrt{2\pi\delta^2}$, where Δ_0 is the mean detuning and $\delta = 1/T^*{}_2$ is the spread in detunings. Calculate the probability P_0 in the same two-pulse Ramsey sequence of part (a) for $T_2^* = 5$ μs. Comment on the result.

(c) Now consider a three-pulse Hahn spin-echo Ramsey sequence:
Consider the following pulse sequence: a fast π/2 pulse around *y*-axis with detuning Δ, free evolution for a time T_A a "time reverse" fast pulse around *x,* free evolution for a time T_B, and then a second fast $\pi/2$ pulse around *y*-axis at the same detuning Δ. The initial state is |1>.



Show, $P_0 = \frac{1}{2}\left(1+\cos[\Delta_0(T_A-T_B)]e^{-\delta^2(T_A-T_B)^2/2}e^{-(T_A+T_B)/T_2}\right)$, and plot for $T_A = 10\mu s$, as a function of T_B=0 to 25 μs.