**Physics 581: Quantum Optics II**
**Problem Set #2**
**Due Thursday Feb. 23, 2022**

**Problem 1: Boson Algebra** (25 points)
This problem is to give you some practice manipulating the boson algebra.

(a) Prove the (over) completeness integral for coherent states

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1} \text{ (Hint: Expand in number states).}$$

This basis is over-complete since as the coherent states are not orthonormal (see next part).

(b) Prove the group property of the displacement operator

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp\{i\,\mathrm{Im}(\alpha\beta^*)\}$$

$$\text{and thus } \langle\alpha|\beta\rangle = e^{-\frac{|\alpha-\beta|^2}{2}} e^{-i\mathrm{Im}(\alpha\beta^*)}$$

(c) Show that the displacement operators are orthogonal according to the Hilbert-Schmidt inner product, $Tr\left(\hat{D}^\dagger(\alpha)\hat{D}(\beta)\right) = \pi\delta^{(2)}(\alpha-\beta)$.

Hint: Recall $Tr\left(\hat{A}\right) = Tr\left(\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha|\hat{A}\right) = \int \frac{d^2\alpha}{\pi}\langle\alpha|\hat{A}|\alpha\rangle$

(d) Show that the displacement operator has the following matrix elements

Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Coherent states: $\langle\alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-|\alpha+\alpha_2-\alpha_1|^2/2} e^{i\,\mathrm{Im}\left(\alpha\alpha_2^* - \alpha_1\alpha^* - \alpha_1\alpha_2^*\right)}$

Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\mathsf{L}_n\left(|\alpha|^2\right)$, where $\mathsf{L}_n$ is the Laguerre polynomial of order $n$

$$L_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{m!} x^m$$

**Problem 2: Ways of calculating the Wigner Function** (25 points)
(a) Show that for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, the Wigner function is

$$W(X,P) = \int_{-\infty}^{\infty} \frac{dY}{2\pi}\, \psi^*\left(X+\frac{Y}{2}\right)\psi\left(X-\frac{Y}{2}\right)e^{-iPY} \text{ , where } W(X,P) = \frac{1}{2}W(\alpha) \ .$$

This the form Wigner originally wrote in terms of the wave function.

(b) Using this show that the Wigner function yields the correct marginals in $X$ and $P$,
$\int_{-\infty}^{\infty} dP\, W(X,P) = |\psi(X)|^2$, $\int_{-\infty}^{\infty} dX\, W(X,P) = |\tilde{\psi}(P)|^2$, and for an arbitrary quadrature

$$\int_{-\infty}^{\infty} dP_\theta\, W(X,P) = |\tilde{\psi}(X_\theta)|^2$$

(c) Fill in the details from lecture and show that

$$W(\alpha) = \frac{2}{\pi^2} e^{2|\alpha|^2} \int d^2\beta \langle -\beta|\hat{\rho}|\beta\rangle e^{-2(\beta\alpha^* - \beta^*\alpha)}$$

**Problem 3: Examples Wigner functions** (25 points)

(a) Fill in the details from lecture and find the Wigner function $W(\alpha)$ and the Husimi function $Q(\alpha)$ for a Fock state $|n\rangle$.

(b) Consider a "cat state"

$$|\psi\rangle = \frac{1}{\mathcal{N}}\left(|\alpha\rangle + e^{i\phi}|-\alpha\rangle\right)$$

- Find the normalization constant $\mathcal{N}$
- Wigner function $W(\alpha)$ and the Husimi function $Q(\alpha)$. Express also as a function of $X$, $P$.
- Plot $W(X,P)$ and $Q(X,P)$, for $\alpha=1$ and $\alpha=3$, and for $\phi=0$ and $\phi=\pi$. Comment on the results.
- For a real $\alpha$ and $\phi=0$, in integrate $W(X,P)$ to show that you obtain the correct marginals.

**Problem 4: Nonclassical light generation via the Kerr effect. (25 points)**
In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description of a single mode is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\kappa}{2}\hat{a}^{\dagger 2}\hat{a}^2 .$$

(a) At $t$=0, suppose we the state a coherent state, $|\alpha_0\rangle$. Show at the time $\kappa t = \pi$, the state becomes a Schrödinger cat, $\left(e^{i\pi/4}|-i\alpha_0\rangle + e^{-i\pi/4}|i\alpha_0\rangle\right)/\sqrt{2}$ .

(Hint: consider $i^n$ for even and odd $n$ and find the periodic pattern).

(b) Show that the Wigner function at time $t$ has the analytic form

$$W(\alpha,t) = \frac{2}{\pi}e^{-2|\alpha|^2}e^{-|\alpha_0|^2}\sum_{n=0}^{\infty}\frac{(2\alpha^*\alpha_0 e^{-i\frac{\kappa t}{2}})^n}{n!}e^{i\frac{\kappa t}{2}n^2}\sum_{m=0}^{\infty}\frac{(2\alpha\alpha_0^* e^{i\frac{\kappa t}{2}})^m}{m!}e^{-i\frac{\kappa t}{2}m^2}e^{-|\alpha_0|^2 e^{-i\kappa t(m-n)}}$$

(c) (Extra credit 10 points)
Take $\alpha$=3. Numerically make a movie of W($\alpha$,t) for $0 \le t \le 2\pi/\kappa$. (you will have to appropriately truncate the Fock space appropriately). Please comment on the results.

(d) Show that the Wigner function evolves in time according to the following partial differential equation

$$\frac{\partial W}{\partial t} = -i\kappa(|\alpha|^2 - 1)\left(\alpha^*\frac{\partial W}{\partial \alpha^*} - \alpha\frac{\partial W}{\partial \alpha}\right) - \frac{i\kappa}{4}\left(\alpha^*\frac{\partial}{\partial \alpha} - \alpha\frac{\partial}{\partial \alpha^*}\right)\frac{\partial^2 W}{\partial\alpha\partial\alpha^*}.$$

What is the equation in the Truncated Wigner Approximation (TWA)? Comment on the physical interpretation.