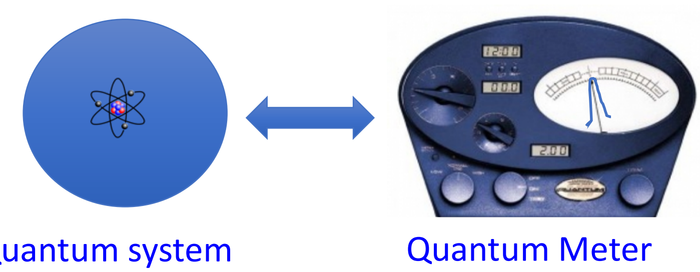
Physics 581: Quantum Optics II
Problem Set #3
Due Thursday Mar. 9, 2023

Problem 1: QND Measurement (35 points)
A “quantum non-demolition” (QND) measurement is of the sort we first learn about in quantum mechanics – we measure an observable and we do not disturb the state if it is in an eigenstate of the observable. Most actual measurements are not of this sort, e.g., when we measure photon-number we typically absorb the photons and they are gone! The standard projective measurement we learn about in which the state of the system collapses to an eigenvector of an observable corresponding the eigenvalue we measure is an example of a QND measurement. It is the limit of a more general POVM which can be realized in the von Neumann paradigm.

A quantum system (S) is coupled to a quantum meter (M). The “pointer” of the meter is initialized in some localized wavepacket $|\Phi_0\rangle$.



The system is then coupled unitarily to the meter in the QND form $\hat{U}_{SM} = e^{-i\chi \hat{A}\otimes\hat{P}}$, where $\hat{A}$ is the observable of the system to be measured, $\hat{P}$ is the momentum operator for the meter, and χ is a coupling constant. We can understand this as the operator that displaces the position of the meter by an amount that proportional to $\hat{A}$.

a) Given an initial state of the system, $|\psi^{in}\rangle = \sum c_a^{in}|a\rangle$, where $|a\rangle$ are the eigenvectors of $\hat{A}$, show that after the interaction, the joint state of the system and meter is the entangled state

$$|\Psi^{out}\rangle_{SM} = e^{-i\chi\hat{A}\otimes\hat{P}}|\psi^{in}\rangle \otimes |\Phi_0\rangle = \sum_a c_a^{in}|a\rangle \otimes |\Phi_{\chi a}\rangle$$

where $|\Phi_{\chi a}\rangle = e^{-i\chi a\hat{P}}|\Phi_0\rangle$

Now if $\langle\Phi_{\chi a'}|\Phi_{\chi a}\rangle = \delta_{aa'}$, the pointer states are distinguishable. In that case by “looking at” the meter will project the meter into the state that correlated with the eigenvalue *a*, and the state of the system will be projected to $|a\rangle$. What about the general case?

b) Suppose we initialize the pointer in a Gaussian wavepacket $\langle X|\Phi_0\rangle = \frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{-\frac{X^2}{4\sigma_0^2}}$. Show that if we do a position measurement of the meter, the (unnormalized) state of the system is

$$|\tilde{\psi}^{out}\rangle = \hat{M}_X|\psi^{in}\rangle$$

where the Kraus (measurement) operator is: $\hat{M}_X = \frac{1}{(2\pi\sigma_0^2)^{1/4}} e^{-\frac{(X-\chi\hat{A})^2}{4\sigma_0^2}}$.

c) We can change variables $\mathcal{A} \equiv X/\chi$, to define the Kraus operator $\hat{M}_{\mathcal{A}} = \frac{1}{(2\pi\sigma_a^2)^{1/4}} e^{-\frac{(\hat{A}-\mathcal{A})^2}{4\sigma_a^2}}$, where $\sigma_0 = \sigma_a/\chi$. Show that

$$\hat{E}_{\mathcal{A}} = \hat{M}_{\mathcal{A}}^\dagger \hat{M}_{\mathcal{A}} = \frac{1}{\sqrt{2\pi\sigma_a^2}} \sum_a e^{-\frac{(a-\mathcal{A})^2}{2\sigma_a^2}} |a\rangle\langle a|$$

and this forms a POVM, i.e., $\int d\mathcal{A}\, \hat{E}_{\mathcal{A}} = \hat{1}$.

d) Show that in the limit $\lim_{\sigma_a \to 0} \hat{E}_{\mathcal{A}} = |a = \mathcal{A}\rangle\langle a = \mathcal{A}|$, a projective measurement.

e) Now suppose the initial state had a Gaussian distribution of amplitudes

$$P^{in}(a) = |c^{in}(a)|^2 = \frac{1}{\sqrt{2\pi\Delta A^2}} e^{\frac{-(a-\langle\hat{A}\rangle)^2}{2\Delta A^2}}$$

Show that according to Quantum Bayes rule, $|\psi^{out}\rangle_{\mathcal{A}} = \frac{\hat{M}_{\mathcal{A}}|\psi^{in}\rangle}{||\hat{M}_{\mathcal{A}}|\psi^{in}\rangle||}$, the output probability distribution of *a* conditioned on having measured $\mathcal{A}$ is

$$P^{out}(a|\mathcal{A}) = \frac{1}{\sqrt{2\pi\Delta A_{out}^2}} e^{\frac{-(a-\langle\hat{A}\rangle_{out})^2}{2\Delta A_{out}^2}}$$

where $\langle A\rangle_{out} = \langle A\rangle_{in} + \frac{\Delta A_{in}^2}{\sigma_a^2 + \Delta A_{in}^2}(\mathcal{A} - \langle A\rangle_{in})$, $\Delta A_{out}^2 = \frac{\sigma_a^2}{\sigma_a^2 + \Delta A_{in}^2}\Delta A_{in}^2$.

Note after the measurement, the uncertainty in *A* is "squeezed" (reduced). This is called measurement induced squeezing. Further show at in the limit $\sigma_a \rightarrow 0$, this posterior probability is the projective measurement.

f) From the Born rule the probability of finding measurement outcome $\mathcal{A}$ conditioned on the state being |*a*> is $P(\mathcal{A}|a) = \langle a|\hat{E}_{\mathcal{A}}|a\rangle$. Calculate this conditional probability and show the *classical* Bayes rule ,

$$P^{out}(a|\mathcal{A}) = \frac{P(\mathcal{A}|a)P^{in}(a)}{||P(\mathcal{A}|a)P^{in}(a)||},$$

where $||P(\mathcal{A}|a)P^{in}(a)|| = \sum_a P(\mathcal{A}|a)P^{in}(a) = \langle\psi^{in}|\hat{E}_{\mathcal{A}}|\psi^{in}\rangle = P_{\mathcal{A}}$

gives the same result as Quantum Bayes rule in part e)!

g) What it is quantum about "quantum backaction" in QND measurement? How is the quantum state "disturbed" in any way differently than Bayesian updating conditioned on the information we learn?

Problem 2: Quantum Channels for Qubits (35 points)

a) Given an input density matrix for a qubit, $\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$, find the output density matrix for the:
(i) depolarizing channel, (ii) dephasing channel, (iii) amplitude damping channel as defined in class.

b) For each of these three channels, use the adjoint map to calculate how the three Paul operators evolve in the Heisenberg picture, $\hat{\sigma}_i^{out} = \mathcal{E}^\dagger[\hat{\sigma}_i]$. From this determine the input-output relation on the components of the Bloch vector $Q_i = \mathrm{tr}[\hat{\rho}\hat{\sigma}_i]$.

c) With the input density matrix written as $\hat{\rho} = \frac{1}{2}\left(\hat{1} + \vec{Q}\cdot\hat{\vec{\sigma}}_i\right)$, using the input-output relation on $\vec{Q}$ to find the output density matrix for each of the three channels. Show that this recovers what you found in part a).

d) The three channels can be seen a Markovian map – the solution to a Lindblad master equation (with no Hamiltonian). Consider first the depolarizing channel. In this case there a three Lindblad "jump" operators $\hat{L}_i = \sqrt{\frac{\Gamma}{3}}\hat{\sigma}_i$.

- Show that the differential Lindblad map, $\hat{\rho}(t+dt) = \mathcal{E}_{dt}[\hat{\rho}(t)]$, is a depolarizing channel with differential probability parameter $dp = \Gamma dt$.
- Show that the corresponding master equation can be written as $\frac{d\hat{\rho}}{dt} = -\frac{4\Gamma}{3}\rho + \frac{2}{3}\hat{1}$.
- Show that the solution to the master equation is $\hat{\rho}(t) = e^{-\frac{4\Gamma}{3}t}\hat{\rho}(0) + (1 - e^{-\frac{4\Gamma}{3}t})\frac{1}{2}\hat{1}$. What is the steady state solution?
- This solution $\hat{\rho}(t) = \mathcal{E}_t[\hat{\rho}(0)]$ is a depolarizing channel. What is the parameter $p(t)$?

e) Consider next the dephasing channel. In this case there is one Lindblad jump operator $\hat{L} = \sqrt{\frac{\Gamma}{2}}\hat{\sigma}_z$.

- Show that the differential Lindblad map, $\hat{\rho}(t+dt) = \mathcal{E}_{dt}[\hat{\rho}(t)]$, is a dephasing channel with differential probability parameter $dp = \Gamma dt$.
- Show that the corresponding master equation can be written as $\frac{d\hat{\rho}}{dt} = -\frac{\Gamma}{2}\hat{\rho} + \frac{\Gamma}{2}\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z$.
- Find the solution for the density matrix at time *t* and the steady state solution.
- This solution $\hat{\rho}(t) = \mathcal{E}_t[\hat{\rho}(0)]$ is a dephasing channel. What is the parameter $p(t)$?

f) Finally consider the amplitude damping channel. The is one Lindblad jump operator $\hat{L} = \sqrt{\Gamma}\hat{\sigma}_-$.

- Show that the differential Lindblad map, $\hat{\rho}(t+dt) = \mathcal{E}_{dt}[\hat{\rho}(t)]$, is an amplitude damping channel with differential probability parameter $dp = \Gamma dt$.
- Show that the corresponding master equation can be written as
  $\frac{d\hat{\rho}}{dt} = -\frac{\Gamma}{2}\left(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right) + \Gamma\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+$.
- Find the solution for the density matrix at time *t* and the steady state solution.
- This solution $\hat{\rho}(t) = \mathcal{E}_t[\hat{\rho}(0)]$ is an amplitude damping channel. What is the parameter $p(t)$?