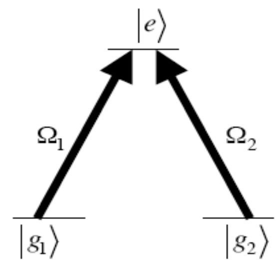
Physics 581, Open Quantum Systems

Problem Set #5

Due: Thursday April 6, 2023

Problem 1: Dark states (25 points)

Let us consider a three level "lambda system"



The two ground states are resonantly coupled to the excited state, each with a different
Rabi frequency. Taking the two ground states as the zero of energy, then in the RWA (and in the rotating frame) the Hamiltonian is

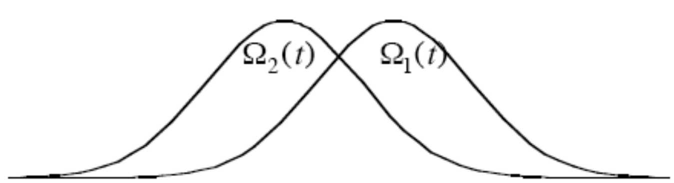
$$\hat{H}_{RF} = \frac{\hbar}{2}\left[\Omega_1\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \Omega_2\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)\right]$$

(a) Find the "dressed states" of this system (i.e. the eigenstates and eigenvalues of the total atom laser system). You should find that one of these states has a **zero** eigenvalue,

$$|\psi_{Dark}\rangle = \frac{\Omega_2|g_1\rangle - \Omega_1|g_2\rangle}{\sqrt{\Omega_1^2+\Omega_2^2}}$$

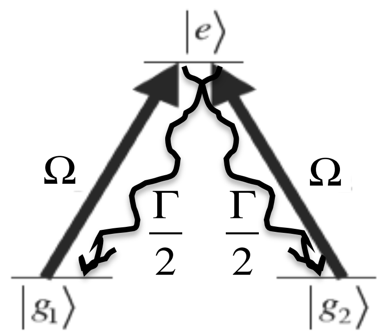
This particular superposition is called a "**dark state**" or uncoupled state because the laser field does not couple it to the excited state.

(b) Adiabatic transfer through the "nonintuitive" pulse sequence. Suppose we want to transfer population from $|g_1\rangle$ to $|g_2\rangle$. A robust method is to use adiabatic passage, *always staying in the local dark state*. This can then be *on resonance*. If we apply a slowly varying pulse $\Omega_2(t)$ overlapped, but followed by $\Omega_1(t)$ shown below, we accomplish this transfer



Sketch the dressed state eigenvalues a function of time. Explain the conditions necessary

to achieve the adiabatic transfer.
(c) When including spontaneous emission of the excited state, the atom will "relax" to the dark state. This is known as coherent population trapping (CPT). Consider, for simplicity, the case that $\Omega_1 = \Omega_2 = \Omega$, and the atom decays with equal rates to the two ground sublevels:



Write the master equation in the basis $\left\{ |D\rangle = \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}}, |B\rangle = \frac{|g_1\rangle + |g_2\rangle}{\sqrt{2}}, |e\rangle \right\}$, where $|D\rangle$ is the dark-state and $|B\rangle$ is the bright-state. Argue that the master equation is

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}^{\dagger}_{eff}\right) + \mathcal{L}_{feed}[\hat{\rho}]$$

$$\hat{H}_{eff} = -i\frac{\hbar\Gamma}{2}|e\rangle\langle e| + \frac{\hbar\sqrt{2}\Omega}{2}\left(|e\rangle\langle B| + |B\rangle\langle e|\right), \quad \mathcal{L}_{feed}[\hat{\rho}] = \frac{\Gamma}{2}\langle e|\hat{\rho}|e\rangle\left(|B\rangle\langle B| + |D\rangle\langle D|\right)$$

(d) Show that the equations of motion for the density matrix in this basis are

$$\dot{\rho}_{ee} = -\Gamma\rho_{ee} + i\frac{\sqrt{2}\Omega}{2}\left(\rho_{eB} - \rho_{Be}\right), \quad \dot{\rho}_{BB} = +\frac{\Gamma}{2}\rho_{ee} - i\frac{\sqrt{2}\Omega}{2}\left(\rho_{eB} - \rho_{Be}\right),$$

$$\dot{\rho}_{eB} = -\frac{\Gamma}{2}\rho_{eB} - i\frac{\sqrt{2}\Omega}{2}\left(\rho_{BB} - \rho_{ee}\right), \quad \dot{\rho}_{DD} = +\frac{\Gamma}{2}\rho_{ee},$$

$$\dot{\rho}_{eD} = -\frac{\Gamma}{2}\rho_{eD} - i\frac{\sqrt{2}\Omega}{2}\rho_{BD}, \quad \dot{\rho}_{BD} = -i\frac{\sqrt{2}\Omega}{2}\rho_{eD}$$

and the steady state solution is $\hat{\rho}^{s.s.} = |D\rangle\langle D|$, i.e., the system relaxes to the dark-state.

Epilogue: The relaxation to the dark state is somewhat mysterious from the equations of motion since a spontaneous decay of along one of the two paths sketched above CANNOT land us in the dark state – we land in $|g_1\rangle$ or $|g_2\rangle$. Actually, we relax to the dark state when we DO NOT see a spontaneous decay. Not seeing spontaneous emission is information too. We'll return to this later when we study "quantum trajectories."

Problem 2: The transfer of coherences in the decay of an oscillator (20 points)

In the simple harmonic oscillator, the decay rate of population in a Fock state |*n*> is proportion to *n*. This makes sense physically, e.g., for a bosonic modes, as the probability of losing a photon will depend on the number of photons in the mode. However, for a coherent state, the rate of decay is independent of the amplitude! We understand this through the "transfer of coherence." Because of the equal spacing of the energy levels, *coherence fed in the master equation*. This transfer of coherences explains the way in which the mean amplitude decays. Let's examine this more closely.

(a) Consider the damped simple harmonic oscillator at finite temperature. Find the equation of motion of the off-diagonal elements (coherences) $\dot{\rho}_{n+1,n}$.

(b) Given $$\langle \hat{a} \rangle = Tr(\hat{a}\hat{\rho}) = \sum_{n=0}^{\infty} \sqrt{n+1}\, \rho_{n+1,n}$$ show that

$$\frac{d}{dt}\langle \hat{a} \rangle = \sum_{n=0}^{\infty} \sqrt{n+1}\, \dot{\rho}_{n+1,n} = (-i\omega - \frac{\Gamma}{2})\langle \hat{a} \rangle$$, independent of the temperature of the bath.

(c) Now consider the *nonlinear* Kerr oscillator, with Hamiltonian

$$\hat{H} = \hbar\omega \hat{a}^{\dagger}\hat{a} + \hbar\kappa \hat{a}^{\dagger 2}\hat{a}^{2}$$

Including the same Lindbladian for the damped SHO at zero temperature, find the equation of motion of $\dot{\rho}_{n+1,n}$, and from this show by the same procedure in (b)

$$\frac{d}{dt}\langle \hat{a} \rangle = (-i\omega - \frac{\Gamma}{2})\langle \hat{a} \rangle - i\kappa \langle \hat{n}\hat{a} \rangle$$

(d) Repeat this calculation by directly finding $d\langle \hat{a} \rangle/dt$ from the commutators.

Thus, for the nonlinear oscillator the transfer of coherences is not perfect as the spacing between levels is not equal. Thus, the decay of the oscillator depends on its amplitude.

(e) But wait a minute! How did we get away with the same Lindbladian? Why are any transfer of coherences included in the first place, since the spacing between levels is not equal. Why didn't we need to include the Kerr Hamiltonian in the derivation of the Lindbladian?

Problem 3: Decoherence and cat states (25 points)

Consider the damped simple harmonic oscillator governed by the master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}(\hat{a}^\dagger\hat{a}\rho + \rho\hat{a}^\dagger\hat{a}) + \Gamma\hat{a}\rho\hat{a}^\dagger \text{, where } \hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}.$$

(a) Suppose we prepare an initial pure coherent state, $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$. From the master equation,

$$\hat{\rho}(dt) = \left(\hat{1} - \frac{i}{\hbar}\hat{H}_{\text{eff}}dt\right)\hat{\rho}(0)\left(\hat{1} + \frac{i}{\hbar}\hat{H}^\dagger_{\text{eff}}dt\right) + \Gamma dt\ \hat{a}\hat{\rho}(0)\hat{a}^\dagger \quad \hat{H}_{\text{eff}} = \hat{H} - i\frac{\hbar\Gamma}{2}\hat{a}^\dagger\hat{a}$$

Show that $\hat{\rho}(dt) = |\alpha e^{-i\omega dt}e^{-\Gamma dt/2}\rangle\langle\alpha e^{-i\omega dt}e^{-\Gamma dt/2}|$ (Hint keep terms only to order *dt*). From this argue that for finite time $\hat{\rho}(t) = |\alpha(t)\rangle\langle\alpha(t)|$, where $\alpha(t) = \alpha e^{-i\omega t}e^{-\Gamma t/2}$, i.e., the state remains a pure coherent state for all times, with a complex amplitude that follows the classical trajectory.

(b) Now consider and initial pure state which is a superposition of coherent states $|\psi(0)\rangle = \mathcal{N}(|\alpha\rangle + |\beta\rangle)$, where $\mathcal{N}^{-2} = 2(1 + \text{Re}\langle\alpha|\beta\rangle)$ is the normalization. Consider the different map generated by the master equation. Show that,

$$\begin{aligned}\mathcal{E}(dt)\left[|\alpha\rangle\langle\beta|\right] &= \left(\hat{1} - \frac{i}{\hbar}\hat{H}_{\text{eff}}dt\right)|\alpha\rangle\langle\beta|\left(\hat{1} + \frac{i}{\hbar}\hat{H}^\dagger_{\text{eff}}dt\right) + \Gamma dt\ \hat{a}|\alpha\rangle\langle\beta|\hat{a}^\dagger \\ &= \langle\beta|\alpha\rangle^{\Gamma dt}\ |\alpha e^{-i\omega dt}e^{-\Gamma dt/2}\rangle\langle\beta e^{-i\omega dt}e^{-\Gamma dt/2}|\end{aligned}$$

(c) From this show that for an initial superposition of coherent states, the solution to the master equation is

$$\hat{\rho}(t) = \mathcal{N}^2\left(|\alpha(t)\rangle\langle\alpha(t)| + f(t)|\alpha(t)\rangle\langle\beta(t)| + f^*(t)|\beta(t)\rangle\langle\alpha(t)| + |\beta(t)\rangle\langle\beta(t)|\right)$$

$$\alpha(t) = \alpha e^{-i\omega t}e^{-\Gamma t/2},\ \beta(t) = \beta e^{-i\omega t}e^{-\Gamma t/2},\ f(t) = \langle\beta|\alpha\rangle^{(1-e^{-\Gamma t})}.$$

(d) Given the result (c) write the Wigner function for the state as a function time.

(e) For a Schrödinger cat state with $\beta = -\alpha$ and $|\alpha| >> 1$, qualitatively describe the time evolution of the state.

(f) (Extra credit 10 points). Make a move of the time evolution of the cat state evolving according to master equation for three cases: α=1,4,10. For simplicity, go into the rotating frame and by setting ω=0. Comment on your results.