Physics 581, Open Quantum Systems

Problem Set #5

Due: Thursday April 20, 2023

Problem 1: Phase space representations for the master equation in different operator orderings (25 points)

We have seen that master equation for the damped simple harmonic oscillator, when expressed in the Wigner-Weyl representation takes the form of a Fokker-Planck equation. Let's consider now other operator ordering.

(a) Show that $|\alpha\rangle\langle\alpha| = e^{\alpha\hat{a}^\dagger}|0\rangle\langle 0|e^{\alpha^*\hat{a}}\ e^{-\alpha^*\alpha}$ and from this show that

$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right)|\alpha\rangle\langle\alpha| \qquad |\alpha\rangle\langle\alpha|\hat{a} = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|$$

(b) From the definition of the Glauber P-representation, $\hat{\rho}(t) = \int d^2\alpha P(\alpha,t)|\alpha\rangle\langle\alpha|$, use part (a) to show that

$$\hat{a}\hat{\rho}(t) \to \alpha P(\alpha,t) \qquad \hat{\rho}(t)\hat{a}^\dagger \to \alpha^* P(\alpha,t)$$
$$\hat{\rho}(t)\hat{a} \to \left(\alpha - \frac{\partial}{\partial\alpha^*}\right)P(\alpha,t) \qquad \hat{a}^\dagger\hat{\rho}(t) \to \left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha,t)$$

and thus, under the Lindblad master equation for the damped SHO, the P-function evolves as

$$\frac{\partial P(\alpha,t)}{\partial t} = \frac{\Gamma}{2}\left(\frac{\partial}{\partial\alpha}\alpha P + \frac{\partial}{\partial\alpha^*}\alpha^* P\right) + \Gamma\bar{n}\frac{\partial^2 P}{\partial\alpha\partial\alpha^*}.$$

Thus, the P-function also satisfies the Fokker-Planck equation, but with a different diffusion coefficient, when compared to the equation we found for the Wigner function.

(c) For an initial coherent state, $|\alpha_0\rangle$, using (b) what is the P-function as a function of time in a zero-temperature reservoir?

(d) Now consider the Husimi *Q*-function, $Q(\alpha,t) = Tr\left(|\alpha\rangle\langle\alpha|\hat{\rho}(t)\right)$. Show that is this reprsentation

$$\hat{a}\hat{\rho}(t) \to \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)Q(\alpha,t) \qquad \hat{\rho}(t)\hat{a}^\dagger \to \left(\alpha^* + \frac{\partial}{\partial\alpha}\right)Q(\alpha,t)$$
$$\hat{\rho}(t)\hat{a} \to \alpha Q(\alpha,t) \qquad \hat{a}^\dagger\hat{\rho}(t) \to \alpha^* Q(\alpha,t)$$

(e) Show that under the Lindblad master equation for the damped SHO, the Q-function evolves as

$$\frac{\partial Q(\alpha,t)}{\partial t} = \frac{\Gamma}{2}\left(\frac{\partial}{\partial \alpha}\alpha Q + \frac{\partial}{\partial \alpha^*}\alpha^* Q\right) + \Gamma(\bar{n}+1)\frac{\partial^2 Q}{\partial\alpha\partial\alpha^*}$$

Thus, we have found that the *s*-ordered quasiprobability evolves according to the Fokker-Planck equation

$$\frac{\partial W_s(\alpha,t)}{\partial t} = \frac{\Gamma}{2}\left(\frac{\partial}{\partial \alpha}\alpha W_s + \frac{\partial}{\partial \alpha^*}\alpha^* W_s\right) + \Gamma(\bar{n}+\frac{s+1}{2})\frac{\partial^2 W_s}{\partial\alpha\partial\alpha^*}$$

where *s* = -1,0,1 for *P*,*Q*,*W* respectively. For initial Gaussian distributions they remain Gaussian, but with a different diffusion coefficient representing the different manner in which quantum fluctuations enter, depending on operator ordering.

(f) Now consider the closed system Kerr system Hamiltonian $\hat{H} = \frac{1}{2}\hbar\kappa\hat{a}^{\dagger 2}\hat{a}^2$.
Show that the equation of motion for the Husimi *Q*-function is

$$\frac{\partial Q}{\partial t} = -i\kappa|\alpha|^2\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)Q(\alpha,t) - i\frac{\kappa}{2}\left(\alpha^{*2}\frac{\partial^2}{\partial\alpha^{*2}} - \alpha^2\frac{\partial^2}{\partial\alpha^2}\right)Q(\alpha,t)$$

The first term is the classical flow of the Kerr effect – rotation in phase space by an angle proportional to the square amplitude. The second term is the nonclassical evolution. But unlike for the Wigner function, for the Husmi the deviation arises from the second order derivative rather than third order.

(g) **Extra credit 10 points**
In comparison to the equation of motion we found for the Wigner function, the Husimi distribution satisfies a Fokker- Planck equation! However, the motion is still not classical.

- Show that the diffusion matrix is not positive semidefinite.
- In the presence of decoherence, under what conditions is the diffusion matrix positive semidefinite and thus classical?

Problem 2: Heisenberg-Langevin Equations for damped Rabi oscillations (25 points)
Consider the problem of a two-level atom driven by a laser field near resonance, and undergoing spontaneous emission. This is the problem of damped Rabi oscillations, equivalent to spin resonance with T_1 and T_2 damping as studied both phenomenologically and in terms of the Lindblad master equation. Let's look at this problem in the Heisenberg-Langevin picture.

(a) Consider the familiar Hamiltonian of the two-level atom (qubit) coupled to a zero temperature bath of simple harmonic oscillators (the electromagnetic vacuum) driven by a laser field. In the rotating frame of the laser and the RWA, the Hamiltonian is

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB}$$

$$\hat{H}_S = -\frac{\hbar\Delta}{2}\sigma_z + \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) \qquad \hat{H}_B = \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k$$

$$\hat{H}_{SB} = \hbar\sum_k (g_k \hat{b}_k \hat{\sigma}_+ e^{i\omega_L t} + g_k^* \hat{\sigma}_- \hat{b}_k^\dagger e^{-i\omega_L t})$$

Show that the Heisenberg equations in the rotating frame of motion are,

$$\frac{d}{dt}\hat{\sigma}_+ = -i\Delta\sigma_+(t) - i\frac{\Omega}{2}\hat{\sigma}_z(t) - i\sum_k g_k^* b_k^\dagger(t)\hat{\sigma}_z(t)e^{-i\omega_L t}$$

$$\frac{d}{dt}\hat{\sigma}_z = -i\Omega(\hat{\sigma}_+(t) - \hat{\sigma}_-(t)) - 2i\sum_k (g_k\hat{b}_k(t)\hat{\sigma}_+(t)e^{i\omega_L t} - g_k^*\hat{\sigma}_-(t)\hat{b}_k^\dagger(t)e^{-i\omega_L t})$$

$$\frac{d}{dt}\hat{b}_k = -i\omega_k\hat{b}_k(t) - ig_k^*\hat{\sigma}_-(t)e^{-i\omega_L t}$$

(b) We can formally integrate the bath modes

$$\hat{b}_k(t) = \hat{b}_k^{vac}(t) + \hat{b}_k^{source}(t)$$

where $\hat{b}_k^{vac}(t) = \hat{b}_k(0)e^{-i\omega_k t}$ is the "vacuum" contribution, and
$\hat{b}_k^{source}(t) = -ig_k^* \int_0^t dt' \hat{\sigma}_-(t)e^{-i\omega_L t'} e^{-i\omega_k(t-t')}$ is the "source contribution" arising from the radiation into the field. Note, operator ordering now matters!

Under the Markov approximation, show that the Heisenberg Langevin equations for the two-level atom in the rotating frame are

$$\frac{d}{dt}\hat{\sigma}_+ = \left(-i\Delta + \frac{\Gamma}{2}\right)\sigma_+(t) - i\frac{\Omega}{2}\hat{\sigma}_z(t) + \hat{\mathcal{F}}^\dagger(t)\hat{\sigma}_z$$

$$\frac{d}{dt}\hat{\sigma}_z = -\Gamma(\sigma_z(t) + \hat{1}) - i\Omega(\hat{\sigma}_+(t) - \hat{\sigma}_-(t)) - 2(\hat{\mathcal{F}}(t)\hat{\sigma}_+(t) + \hat{\sigma}_-(t)\hat{\mathcal{F}}^\dagger(t))$$

$$\hat{\mathcal{F}}(t) = i\sum_k g_k\hat{b}_k(0)e^{-i(\omega_k-\omega_L)t}$$

(c) With the components of the Bloch vector components, $\mathbf{Q} = (u, v, w) = (\langle\hat{\sigma}_x\rangle, \langle\hat{\sigma}_y\rangle, \langle\hat{\sigma}_z\rangle)$, tracing over the system and bath show that we recover the usual damped Bloch equations as given in Lecture 4.

(d) Use the Heisenberg Langevin equations in (b) to find the equation of motion for operator $\hat{\sigma}_+\hat{\sigma}_-$ and show that this agrees with the equation of motion for $\hat{\sigma}_z$.